A Review:
Operators involving ∇ in Curvilinear Orthogonal Coordinates

and

∇ Vector Identities and Product Rules

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Contents

Abstract

This document provides easily understandable derivations for all important ∇ operators applicable to arbitrary orthogonal coordinate systems in $\mathbb{R}^3$ (like spherical, cylindrical, elliptic, parabolic, hyperbolic,..). These include the laplacian (of a scalar function and a vector field), the gradient and its absolute value (of a scalar function), the divergence (of a vector field), the curl (of a vector field) and the convective operator (acting on a scalar function and a vector field) (**Section 3**). For the first two mentioned special coordinate systems the operators are calculated using these expressions along with some remarks on their simplifications in cartesian coordinates (**Section 4**). Alongside these calculations, some useful derivations are presented, including the volume element and the velocity components. The Laplacian as well as the absolute value of the gradient are also calculated the clumsy way (ab)using the chain-rule (**Section 1**). Using the same tedious way as in section 1, the angular momentum operators arising in quantum mechanics are derived for spherical coodinates (**Section 2**) only. Furthermore, all important product rules for the ∇ operator, which are utilized throughout the derivations in section 3, are proven using the Levi-Civita-Symbol and the Kronecker-Delta (**Section 5**).

1 The Laplacian in Spherical Coordinates - The Tedious Way

In this section the Laplacian operator acting on a scalar function is derived for the special case of spherical coordinates only. The way presented here and often described in physic books is very tedious and involved, though only the very basic concepts of partial differential calculus are used here. However, the operator may be calculated *way* easier using the method decribed in (Section 3) and carried out in (Section 4)!
Recall the definition of the Laplacian in cartesian coordonates:

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{1}$$

Some things we need and the transforms:

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{2}$$

$$\theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad \phi = \arctan\left(\frac{y}{x}\right) \qquad r = \sqrt{x^2+y^2+z^2} \tag{3}$$

$$f = f\left(x\left(r,\theta,\phi\right), y\left(r,\theta,\phi\right), z\left(r,\theta,\phi\right)\right) \tag{4}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} \tag{5}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} \tag{6}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z} \tag{7}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \qquad \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}} \qquad (\arctan x)' = \frac{1}{1+x^2} \qquad \left(\arctan\left(\frac{1}{x}\right)\right)' = -\frac{1}{1+x^2} \tag{8}$$

$$\left(\arctan\left(\frac{y}{x}\right)\right)'_y = -\frac{y}{y^2+x^2} \qquad \left(\arctan\left(\frac{y}{x}\right)\right)'_x = \frac{x}{x^2+y^2} \qquad \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} \tag{9}$$

Now we begin:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\left[\frac{1}{2}\frac{2r\sin\theta\cos\phi}{r}\right] + \frac{\partial f}{\partial \theta}\left[-\frac{1}{\sqrt{1-\frac{r^2\cos^2\theta}{r^2}}}z\frac{-1}{2}\frac{2x}{\left(r^2\right)^{3/2}}\right] + \frac{\partial f}{\partial \phi}\left[-\frac{r\sin\theta\sin\phi}{r^2\left(\sin^2\theta\left(\cos^2\phi+\sin^2\phi\right)\right)}\right] \\
&= \frac{\partial f}{\partial r}\sin\theta\cos\phi + \frac{\partial f}{\partial \theta}\cos\theta\cos\phi\frac{1}{r} - \frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\left[\frac{1}{2}\frac{2r\sin\theta\sin\phi}{r}\right] + \frac{\partial f}{\partial \theta}\left[-\frac{1}{\sqrt{1-\frac{r^2\cos^2\theta}{r^2}}}z\frac{-1}{2}\frac{2y}{\left(r^2\right)^{3/2}}\right] + \frac{\partial f}{\partial \phi}\left[-\frac{r\sin\theta\cos\phi}{r^2\left(\sin^2\theta\left(\cos^2\phi+\sin^2\phi\right)\right)}\right] \\
&= \frac{\partial f}{\partial r}\sin\theta\sin\phi + \frac{\partial f}{\partial \theta}\cos\theta\sin\phi\frac{1}{r} + \frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial r}\left[\frac{1}{2}\frac{2r\cos\theta}{r}\right] + \frac{\partial f}{\partial \theta}\left[-\frac{1}{\sin\theta}\frac{r - r\cos\theta\frac{1}{2r}2r\cos\theta}{r^2}\right] + \frac{\partial f}{\partial \phi}\left[0\right] \\
&= \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}
\end{aligned} \tag{12}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\left(\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\right)\frac{\partial r}{\partial x}+\left(\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)\right)\frac{\partial \theta}{\partial x}+\left(\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial x}\right)\right)\frac{\partial \phi}{\partial x} \tag{13}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\left(\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\right)\frac{\partial r}{\partial x}+\left(\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)\right)\frac{\partial \theta}{\partial y}+\left(\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial y}\right)\right)\frac{\partial \phi}{\partial y} \tag{14}$$

$$\frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)=\left(\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right)\right)\frac{\partial r}{\partial z}+\left(\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial z}\right)\right)\frac{\partial \theta}{\partial z}+\left(\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial z}\right)\right)\frac{\partial \phi}{\partial z} \tag{15}$$

Plugging equations (10),(11),(12) into (13),(14),(15) we get:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\cos\theta\cos\phi\frac{1}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right]\sin\theta\cos\phi+\ldots\\
&\quad \frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\cos\theta\cos\phi\frac{1}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right]\frac{\cos\theta\cos\phi}{r}+\ldots\\
&\quad \frac{\partial}{\partial \phi}\left[\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial \theta}\cos\theta\cos\phi\frac{1}{r}-\frac{\partial f}{\partial \phi}\frac{\sin\phi}{r\sin\theta}\right]\frac{-\sin\phi}{r\sin\theta}\\
&= \left[\sin\theta\cos\phi\frac{\partial^2 f}{\partial r^2}-\cos\theta\cos\phi\frac{1}{r^2}\frac{\partial f}{\partial \theta}+\frac{\sin\phi}{r^2\sin\theta}\frac{\partial f}{\partial \phi}\right]\sin\theta\cos\phi+\ldots\\
&\quad \left[\cos\theta\cos\phi\frac{\partial f}{\partial r}-\sin\theta\cos\phi\frac{1}{r}\frac{\partial f}{\partial \theta}+\cos\theta\cos\phi\frac{1}{r}\frac{\partial^2 f}{\partial \theta^2}+\frac{\cos\theta\sin\phi}{r\sin^2\theta}\frac{\partial f}{\partial \phi}\right]\frac{\cos\theta\cos\phi}{r}+\ldots\\
&\quad \left[-\sin\theta\sin\phi\frac{\partial f}{\partial r}-\cos\theta\sin\phi\frac{1}{r}\frac{\partial f}{\partial \theta}-\frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}-\frac{\sin\phi}{r\sin\theta}\frac{\partial^2 f}{\partial \phi^2}\right]\frac{-\sin\phi}{\sin\theta}\\
&= \sin^2\theta\cos^2\phi\frac{\partial^2 f}{\partial r^2}-\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial f}{\partial \theta}+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial f}{\partial \phi}+\ldots\\
&\quad \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial f}{\partial r}-\frac{\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial f}{\partial \theta}+\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{\cos^2\theta\cos\phi\sin\phi}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi}+\ldots\\
&\quad \frac{\sin^2\phi}{r}\frac{\partial f}{\partial r}+\frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial f}{\partial \theta}+\frac{\cos\phi\sin\phi}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi}+\frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial r}\sin\theta\sin\phi+\frac{\partial f}{\partial \theta}\cos\theta\sin\phi\frac{1}{r}+\frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}\right]\sin\theta\sin\phi+\ldots\\
&\quad \frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial r}\sin\theta\sin\phi+\frac{\partial f}{\partial \theta}\cos\theta\sin\phi\frac{1}{r}+\frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}\right]\frac{\cos\theta\sin\phi}{r}+\ldots\\
&\quad \frac{\partial}{\partial \phi}\left[\frac{\partial f}{\partial r}\sin\theta\sin\phi+\frac{\partial f}{\partial \theta}\cos\theta\sin\phi\frac{1}{r}+\frac{\partial f}{\partial \phi}\frac{\cos\phi}{r\sin\theta}\right]\frac{\cos\phi}{r\sin\theta}\\
&= \sin^2\theta\sin^2\phi\frac{\partial^2 f}{\partial r^2}-\frac{\cos\theta\sin\theta\sin^2\phi}{r^2}\frac{\partial f}{\partial \theta}-\frac{\cos\phi\sin\phi}{r^2}\frac{\partial f}{\partial \phi}+\ldots\\
&\quad \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial f}{\partial r}-\frac{\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial f}{\partial \theta}+\frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 f}{\partial \theta^2}-\frac{\cos\phi\cos^2\theta\sin\phi}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi}+\ldots\\
&\quad \frac{\cos^2\phi}{r}\frac{\partial f}{\partial r}+\frac{\cos^2\phi\cos\theta}{r\sin\theta}\frac{\partial f}{\partial \theta}-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi}+\frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^2} &= \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right]\cos\theta+\ldots\\
&\quad \frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right]\frac{-\sin\theta}{r}\\
&= \cos^2\theta\frac{\partial^2 f}{\partial r^2}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta}+\ldots\\
&\quad \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}+\frac{\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial \theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}
\end{aligned} \tag{18}$$

Using equation (16),(17),(18) we obtain the Laplacian:

$$\begin{aligned}
\Delta f &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial^2 f}{\partial r^2}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial f}{\partial r}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta}+\ldots \\
&\quad \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}+\frac{\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} \\
&= \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta} \\
&= \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial^2 f}{\partial\theta^2}+\cos\theta\frac{\partial f}{\partial\theta}\right) \\
&= \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)
\end{aligned}$$

$$\Delta = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{19}$$

We also note the following relations:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)=\frac{1}{r^2}\left(2r\frac{\partial f}{\partial r}+r^2\frac{\partial^2 f}{\partial r^2}\right)=\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial r^2} \tag{20}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf(r))=\frac{1}{r}\frac{\partial}{\partial r}\left(f(r)+r\frac{\partial f}{\partial r}\right)=\frac{1}{r}\left(\frac{\partial f}{\partial r}+\frac{\partial f}{\partial r}+r\frac{\partial^2 f}{\partial r^2}\right)=\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial r^2} \tag{21}$$

And hence obtain the Laplacian operator in spherical coordinates:

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{22}$$

Or equivalently:

$$\Delta=\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{23}$$

The absolute value of the gradient in cartesian coordinates is:

$$|\nabla f|^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2 \tag{24}$$

Using equations (10),(11),(12) and $(x+y+z)^2=x^2+y^2+z^2+2xy+2xz+2yz$ we get:

$$\begin{aligned}
|\nabla f|^2 &= \left(\frac{\partial f}{\partial r}\sin\theta\cos\phi+\frac{\partial f}{\partial\theta}\cos\theta\cos\phi\frac{1}{r}-\frac{\partial f}{\partial\phi}\frac{\sin\phi}{r\sin\theta}\right)^2+\left(\frac{\partial f}{\partial r}\sin\theta\sin\phi+\frac{\partial f}{\partial\theta}\cos\theta\sin\phi\frac{1}{r}+\frac{\partial f}{\partial\phi}\frac{\cos\phi}{r\sin\theta}\right)^2+\ldots \\
&\quad \left(\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial\theta}\frac{\sin\theta}{r}\right)^2 \\
&= \sin^2\theta\cos^2\phi\left(\frac{\partial f}{\partial r}\right)^2+\cos^2\theta\cos^2\phi\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{\sin^2\phi}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2+\ldots \\
&\quad \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial\theta}-\frac{2\cos\phi\sin\phi}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial\phi}-\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial\phi}+\ldots \\
&\quad \sin^2\theta\sin^2\phi\left(\frac{\partial f}{\partial r}\right)^2+\cos^2\theta\sin^2\phi\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{\cos^2\phi}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2+\ldots \\
&\quad \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial\theta}+\frac{2\sin\phi\cos\phi}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial\phi}+\frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial\phi}+\ldots \\
&\quad \cos^2\theta\left(\frac{\partial f}{\partial r}\right)^2-2\frac{\cos\theta\sin\theta}{r}\frac{\partial f}{\partial r}\frac{\partial f}{\partial\theta}+\frac{\sin^2\theta}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2 \\
&= \left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2
\end{aligned} \tag{25}$$

2 The Angular Momentum Operators in Spherical Coordinates

Using the same technique we calculate the spherical angular momentum operators:

$$\widehat{r}\times\widehat{p}=\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}p_x\\p_y\\p_z\end{pmatrix}=-i\hbar\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}\\\frac{\partial}{\partial z}\end{pmatrix}$$

$$\begin{aligned}
\frac{1}{-i\hbar}\widehat{L_x} &= y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\\
&= r\sin\theta\sin\phi\left[\frac{\partial}{\partial r}\cos\theta-\frac{\partial}{\partial\theta}\frac{\sin\theta}{r}\right]-r\cos\theta\left[\frac{\partial}{\partial r}\sin\theta\sin\phi+\frac{\partial}{\partial\theta}\cos\theta\sin\phi\frac{1}{r}+\frac{\partial}{\partial\phi}\frac{\cos\phi}{r\sin\theta}\right]\\
&= r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r}-\frac{r\sin^2\theta\sin\phi}{r}\frac{\partial}{\partial\theta}-r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r}-\cos^2\theta\sin\phi\frac{r}{r}\frac{\partial}{\partial\theta}-\frac{r\cos\theta\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\\
&= -\sin\phi\frac{\partial}{\partial\theta}-\cot\theta\cos\phi\frac{\partial}{\partial\phi}\\
\widehat{L_x} &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta}-\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) && (26)\\
\frac{1}{-i\hbar}\widehat{L_y} &= z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\\
&= r\cos\theta\left[\frac{\partial}{\partial r}\sin\theta\cos\phi+\frac{\partial}{\partial\theta}\cos\theta\cos\phi\frac{1}{r}-\frac{\partial}{\partial\phi}\frac{\sin\phi}{r\sin\theta}\right]-r\sin\theta\cos\phi\left[\frac{\partial}{\partial r}\cos\theta-\frac{\partial}{\partial\theta}\frac{\sin\theta}{r}\right]\\
&= \cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\\
\widehat{L_y} &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) && (27)\\
\frac{1}{-i\hbar}\widehat{L_z} &= x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\\
&= r\sin\theta\cos\phi\left[\frac{\partial}{\partial r}\sin\theta\sin\phi+\frac{\partial}{\partial\theta}\cos\theta\sin\phi\frac{1}{r}+\frac{\partial}{\partial\phi}\frac{\cos\phi}{r\sin\theta}\right]-\dots\\
&\quad r\sin\theta\sin\phi\left[\frac{\partial}{\partial r}\sin\theta\cos\phi+\frac{\partial}{\partial\theta}\cos\theta\cos\phi\frac{1}{r}-\frac{\partial}{\partial\phi}\frac{\sin\phi}{r\sin\theta}\right]\\
&= r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r}+\sin\theta\cos\phi\cos\theta\sin\phi\frac{\partial}{\partial\theta}+\cos^2\phi\frac{\partial}{\partial\phi}-\dots\\
&\quad \sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r}-\sin\theta\sin\phi\cos\theta\cos\phi\frac{\partial}{\partial\theta}+\sin^2\phi\frac{\partial}{\partial\phi}\\
&= \frac{\partial}{\partial\phi}\\
\widehat{L_z} &= -i\hbar\frac{\partial}{\partial\phi} && (28)
\end{aligned}$$

$$\begin{aligned}
\Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} && (29)\\
|\nabla f|^2 &= \left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2 && (30)\\
\widehat{L_x} &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) && (31)\\
\widehat{L_y} &= i\hbar\left(\cot\theta\sin\phi\frac{\partial}{\partial\phi}-\cos\phi\frac{\partial}{\partial\theta}\right) && (32)\\
\widehat{L_z} &= -i\hbar\frac{\partial}{\partial\phi} && (33)
\end{aligned}$$

3 ∇ Operators in Arbitrary Orthogonal Cuvilinear Coordinate Systems

The Laplacian[1] (of a scalar function and a vector field), the gradient and its absolute value (of a scalar function), the divergence (of a vector field), the curl (of a vector field) and the convective operator are often tabulated in appendices for special chosen ortho-curvilinear coordinate systems like polar and spherical coordinates. In this section we will derive a general expression for all the operators mentioned for any curvilinear orthogonal coordinate system in three dimensions using only basic concepts of algebra and calculus. (No Christoffel Symbols necessary)
We will start with some simple observations:
The chain rule gives:

$$\begin{aligned} \mathrm{d}x &= \frac{\partial x}{\partial q_1}\mathrm{d}q_1 + \frac{\partial x}{\partial q_2}\mathrm{d}q_2 + \frac{\partial x}{\partial q_3}\mathrm{d}q_3 \\ \mathrm{d}y &= \frac{\partial y}{\partial q_1}\mathrm{d}q_1 + \frac{\partial y}{\partial q_2}\mathrm{d}q_2 + \frac{\partial y}{\partial q_3}\mathrm{d}q_3 \\ \mathrm{d}z &= \frac{\partial z}{\partial q_1}\mathrm{d}q_1 + \frac{\partial z}{\partial q_2}\mathrm{d}q_2 + \frac{\partial z}{\partial q_3}\mathrm{d}q_3 \end{aligned}$$

This gives the square of the distance in curvilinear coordinates between two points using $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$:

$$\begin{aligned} (\mathrm{d}s)^2 &= (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \\ &= Q_{11}^2(\mathrm{d}q_1)^2 + Q_{22}^2(\mathrm{d}q_2)^2 + Q_{33}^2(\mathrm{d}q_3)^2 + 2Q_{12}\mathrm{d}q_1\mathrm{d}q_2 + 2Q_{13}\mathrm{d}q_1\mathrm{d}q_3 + 2Q_{32}\mathrm{d}q_3\mathrm{d}q_2 \end{aligned} \tag{34}$$

With the following notation (related to the scale factors Q_i of the coordinate system):

$$Q_{ij}^2 = \left(\frac{\partial x}{\partial q_i}\right)\left(\frac{\partial x}{\partial q_j}\right) + \left(\frac{\partial y}{\partial q_i}\right)\left(\frac{\partial y}{\partial q_j}\right) + \left(\frac{\partial z}{\partial q_i}\right)\left(\frac{\partial z}{\partial q_j}\right) \tag{35}$$

But we know for any orthogonal system of vectors $\{\mathbf{e_k}, k = 1\dots n\}, \quad e_k \cdot e_l = \delta_{lk}$ that the Pythagorean Theorem holds:

$$\left\|\sum_{k=1}^{n}\mathbf{e_k}\right\|^2 = \sum_{k=1}^{n}\|\mathbf{e_k}\|^2$$

And hence only the first three terms of equation (34) contribute and $Q_{ij} = 0$ for $i \neq j$
If $\mathrm{d}q_i \neq 0$ and $\mathrm{d}q_j = 0$ for all $j \neq i$, i.e. when only one of the q_i coordinates changes we have:

$$\mathrm{d}s =: \mathrm{d}s_i = Q_{ii}\mathrm{d}q_i := Q_i\mathrm{d}q_i \qquad Q_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2} \tag{36}$$

Now we write down the **gradient**, realizing that any orthogonal coordinate system is cartesian in a infitisimal neighbourhood of a given point and $\{\mathbf{e_k}, k = 1\dots 3\}$ are the unit vectors associated with this local coordinate system near the given point:

$$\begin{aligned} \nabla f &= \mathbf{e_1}\frac{\partial f}{\partial s_1} + \mathbf{e_2}\frac{\partial f}{\partial s_2} + \mathbf{e_3}\frac{\partial f}{\partial s_3} \\ &= \mathbf{e_1}\frac{\partial f}{Q_1\partial q_1} + \mathbf{e_2}\frac{\partial f}{Q_2\partial q_2} + \mathbf{e_3}\frac{\partial f}{Q_3\partial q_3} \\ &= \sum_{i=1}^{3}\mathbf{e_i}\frac{\partial f}{Q_i\partial q_i} \end{aligned} \tag{37}$$

[1] proof for Laplacian of a scalar function, for the divergence and gradient are outlined in [1], based on [3]: H. Margenau and G. M. Murphy, 'The Mathematics of Physics and Chemistry', D. Van Nostrand Co., Princeton, 1956, pp. 150-175

To see that this expression is correct we will derive it yet another, though similar way (outlined in [6]). First we formally write down the unit vectors at some point in space defined by the curvilinear coordinates $q=(q_1,q_2,q_3)$ by the following equation:

$$\mathbf{e_{q_i}} =: \mathbf{e_i} = \frac{\frac{\partial \mathbf{r}(q)}{\partial q_i}}{\left\|\frac{\partial \mathbf{r}(q)}{\partial q_i}\right\|} = \frac{1}{Q_i}\frac{\partial \mathbf{r}(q)}{\partial q_i} \qquad \Leftrightarrow \qquad \frac{\partial \mathbf{r}(q)}{\partial q_i} = Q_i \mathbf{e_i}$$

With the so-called **scale factors** $\quad Q_i = \left\|\frac{\partial \mathbf{r}(q)}{\partial q_i}\right\|$ (They scale the coordinate vectors to unit length)
These scale factors (in the literature often denoted by h_i!) coincide with our previous definition:

$$Q_i = \left\|\frac{\partial \mathbf{r}(q)}{\partial q_i}\right\| = \left\|\frac{\partial \mathbf{r}(x(q),y(q),z(q))}{\partial q_i}\right\| = \left\|\begin{pmatrix} \frac{\partial x(q)}{\partial q_i} \\ \frac{\partial y(q)}{\partial q_i} \\ \frac{\partial z(q)}{\partial q_i} \end{pmatrix}\right\| = \left(\left(\frac{\partial x(q)}{\partial q_i}\right)^2 + \left(\frac{\partial y(q)}{\partial q_i}\right)^2 + \left(\frac{\partial z(q)}{\partial q_i}\right)^2\right)^{1/2}$$

The unit vectors $\mathbf{e_i}$ point in the direction of growing q_i along the q_i-coordinate line.

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}(q)}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \mathbf{r}(q)}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \mathbf{r}(q)}{\partial q_3}\mathrm{d}q_3 = Q_1\mathrm{d}q_1\mathbf{e_1} + Q_2\mathrm{d}q_2\mathbf{e_2} + Q_3\mathrm{d}q_3\mathbf{e_3} \tag{38}$$

Now we will construct the gradient of a scalar function $f = f(q)$ with unknown components $g_i, \quad i = 1,2,3$. We first note the following facts and use total differentiation:

$$\begin{aligned} \nabla f &= g_1\mathbf{e_1} + g_2\mathbf{e_2} + g_3\mathbf{e_3} \\ \mathrm{d}\mathbf{r} &= Q_1\mathrm{d}q_1\mathbf{e_1} + Q_2\mathrm{d}q_2\mathbf{e_2} + Q_3\mathrm{d}q_3\mathbf{e_3} \\ \mathrm{d}f &= \nabla f\cdot\mathrm{d}\mathbf{r} = Q_1g_1\mathrm{d}q_1 + Q_2g_2\mathrm{d}q_2 + Q_3g_3\mathrm{d}q_3 && (39) \\ \mathrm{d}f &= \frac{\partial f}{\partial q_1}\mathrm{d}q_1 + \frac{\partial f}{\partial q_2}\mathrm{d}q_2 + \frac{\partial f}{\partial q_3}\mathrm{d}q_3 && (40) \end{aligned}$$

Where the orthogonality was used in equation (39). Now, by comparison of (39), (40) we identify the g_i's:

$$Q_i g_i \stackrel{!}{=} \frac{\partial f}{\partial q_i} \quad \Rightarrow \quad g_i = \frac{\partial f}{Q_i \partial q_i}$$

Hence we reconstruct the gradient by plugging in the formely unknown coefficients:

$$\nabla f = \frac{\mathbf{e_1}}{Q_1}\frac{\partial f}{\partial q_1} + \frac{\mathbf{e_2}}{Q_2}\frac{\partial f}{\partial q_2} + \frac{\mathbf{e_3}}{Q_3}\frac{\partial f}{\partial q_3} = \sum_{i=1}^{3}\mathbf{e_i}\frac{\partial f}{Q_i\partial q_i}$$

...which is exactly the equation we derived before (37)!
From now on we will proceed in a straight-forward manner to get all the other vector operators involving ∇ using several tricks, utilizing the right-handedness and orthonormality of the coordinate system for example. The first thing we will find is the **divergence** expression which is necessary in order to find the Laplacian. Here a simple product rule is used (proof in appendix):

$$\nabla\cdot\mathbf{g} = \nabla\cdot(\mathbf{e_1}g_1 + \mathbf{e_2}g_2 + \mathbf{e_3}g_3) = \sum_{i=1}^{3}\nabla\cdot(\mathbf{e_i}g_i) = \sum_{i=1}^{3}(\mathbf{e_i}\cdot\nabla g_i + g_i\nabla\cdot\mathbf{e_i}) \tag{41}$$

The first term in the sum can be written the following way using eqn. (37) and the orthogonality of the coordinate system:

$$\mathbf{e_i}\cdot\nabla g_i = \mathbf{e_i}\cdot\left(\mathbf{e_1}\frac{\partial g_i}{Q_1\partial q_1} + \mathbf{e_2}\frac{\partial g_i}{Q_2\partial q_2} + \mathbf{e_3}\frac{\partial g_i}{Q_3\partial q_3}\right) = \frac{\partial g_i}{Q_i\partial q_i} \tag{42}$$

The last term in the sum in eqn (41) is not equal to zero for non-cartesian coordinate systems!
We will need some preparing steps in order to compute $\nabla \cdot \mathbf{e_i}$. Using equation (37) with $f = q_i$ we get:

$$\nabla q_i = \mathbf{e_1} \frac{\partial q_i}{Q_1 \partial q_1} + \mathbf{e_2} \frac{\partial q_i}{Q_2 \partial q_2} + \mathbf{e_3} \frac{\partial q_i}{Q_3 \partial q_3} = \mathbf{e_i} \frac{1}{Q_i} \tag{43}$$

We also need:

$$\nabla \cdot \mathbf{e_1} = \nabla \cdot (\mathbf{e_2} \times \mathbf{e_3}) = \mathbf{e_3} \cdot \nabla \times \mathbf{e_2} - \mathbf{e_2} \cdot \nabla \times \mathbf{e_3} \tag{44}$$

Where the right handedness of the system was used and the vector identity (or product rule) $\nabla \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \nabla \times \mathbf{v} - \mathbf{v} \cdot \nabla \times \mathbf{w}$ as well (see appendix for proof).
We will furthermore encounter the following expression which we derive now:

$$\nabla \frac{1}{Q_1} = \mathbf{e_1} \frac{\partial \frac{1}{Q_1}}{Q_1 \partial q_1} + \mathbf{e_2} \frac{\partial \frac{1}{Q_1}}{Q_2 \partial q_2} + \mathbf{e_3} \frac{\partial \frac{1}{Q_1}}{Q_3 \partial q_3} = -\mathbf{e_1} \frac{\partial Q_1}{Q_1^3 \partial q_1} - \mathbf{e_2} \frac{\partial Q_1}{Q_1^2 Q_2 \partial q_2} - \mathbf{e_3} \frac{\partial Q_1}{Q_1^2 Q_3 \partial q_3} \tag{45}$$

Now we begin with the following trick and use $\nabla \times \nabla f = 0$ and the product rule:

$$\nabla \times \nabla q_1 = 0 = \nabla \times \left(\mathbf{e_1} \frac{1}{Q_1} \right) = \nabla \frac{1}{Q_1} \times \mathbf{e_1} + \frac{1}{Q_1} \nabla \times \mathbf{e_1} = -\mathbf{e_1} \times \nabla \frac{1}{Q_1} + \frac{1}{Q_1} \nabla \times \mathbf{e_1} \tag{46}$$

This gives after rearranging:

$$\nabla \times \mathbf{e_1} = Q_1 \mathbf{e_1} \times \nabla \frac{1}{Q_1} \tag{47}$$

Using equation (47) and (45) we get:

$$\begin{aligned} \nabla \times \mathbf{e_1} &= Q_1 \mathbf{e_1} \times \left(-\mathbf{e_1} \frac{\partial 1}{Q_1^3 \partial q_1} - \mathbf{e_2} \frac{\partial Q_1}{Q_1^2 Q_2 \partial q_2} - \mathbf{e_3} \frac{\partial Q_1}{Q_1^2 Q_3 \partial q_3} \right) \\ &= -\frac{\mathbf{e_1} \times \mathbf{e_2}}{Q_1 Q_2} \frac{\partial Q_1}{\partial q_2} - \frac{\mathbf{e_1} \times \mathbf{e_3}}{Q_1 Q_3} \frac{\partial Q_1}{\partial q_3} \\ &= -\frac{\mathbf{e_3}}{Q_1 Q_2} \frac{\partial Q_1}{\partial q_2} + \frac{\mathbf{e_2}}{Q_1 Q_3} \frac{\partial Q_1}{\partial q_3} \end{aligned} \tag{48}$$

Using equation (44) and appropriate cyclic permutations of the previous equation we have:

$$\begin{aligned} \nabla \cdot \mathbf{e_1} &= \mathbf{e_3} \cdot (\nabla \times \mathbf{e_2}) - \mathbf{e_2} \cdot (\nabla \times \mathbf{e_3}) \\ &= \mathbf{e_3} \cdot \left(-\frac{\mathbf{e_1}}{Q_2 Q_3} \frac{\partial Q_2}{\partial q_3} + \frac{\mathbf{e_3}}{Q_1 Q_2} \frac{\partial Q_2}{\partial q_1} \right) - \mathbf{e_2} \cdot \left(-\frac{\mathbf{e_2}}{Q_1 Q_3} \frac{\partial Q_3}{\partial q_1} + \frac{\mathbf{e_1}}{Q_2 Q_3} \frac{\partial Q_3}{\partial q_2} \right) \\ &= \frac{1}{Q_1 Q_3} \frac{\partial Q_3}{\partial q_1} + \frac{1}{Q_1 Q_2} \frac{\partial Q_2}{\partial q_1} = \frac{1}{Q_1 Q_2 Q_3} \left(Q_3 \frac{\partial Q_2}{\partial q_1} + Q_2 \frac{\partial Q_3}{\partial q_1} \right) = \frac{1}{Q_1 Q_2 Q_3} \left(\frac{\partial (Q_2 Q_3)}{\partial q_1} \right) \end{aligned} \tag{49}$$

Or in general:

$$\nabla \cdot \mathbf{e_i} = \frac{1}{Q_1 Q_2 Q_3} \left(\frac{\partial \left(\frac{Q_1 Q_2 Q_3}{Q_i} \right)}{\partial q_i} \right) \tag{50}$$

Now we write down $\nabla \cdot \mathbf{g}$:

$$\begin{aligned} \nabla \cdot \mathbf{g} &= g_1 \nabla \cdot \mathbf{e_1} + g_2 \nabla \cdot \mathbf{e_2} + g_3 \nabla \cdot \mathbf{e_3} + \mathbf{e_1} \cdot \nabla g_1 + \mathbf{e_2} \cdot \nabla g_2 + \mathbf{e_3} \cdot \nabla g_3 \\ &= \frac{g_1}{Q_1 Q_2 Q_3} \left(\frac{\partial (Q_2 Q_3)}{\partial q_1} \right) + \frac{g_2}{Q_1 Q_2 Q_3} \left(\frac{\partial (Q_1 Q_3)}{\partial q_2} \right) + \frac{g_3}{Q_1 Q_2 Q_3} \left(\frac{\partial (Q_1 Q_2)}{\partial q_3} \right) + \frac{\partial g_1}{Q_1 \partial q_1} + \frac{\partial g_2}{Q_2 \partial q_2} + \frac{\partial g_3}{Q_3 \partial q_3} \\ &= \frac{1}{Q_1 Q_2 Q_3} \left[\frac{\partial \left(g_1 Q_2 Q_3 \right)}{\partial q_1} + \frac{\partial \left(g_2 Q_1 Q_3 \right)}{\partial q_2} + \frac{\partial \left(g_3 Q_1 Q_2 \right)}{\partial q_3} \right] = \frac{1}{Q_1 Q_2 Q_3} \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \left(Q_1 Q_2 Q_3 \frac{g_i}{Q_i} \right) \end{aligned} \tag{51}$$

If we now set $\mathbf{g} = \nabla f$ we get the Laplacian in curvilinear orthogonal coordinate systems:

$$g = \nabla f = \mathbf{e_1} \frac{\partial f}{Q_1 \partial q_1} + \mathbf{e_2} \frac{\partial f}{Q_2 \partial q_2} + \mathbf{e_3} \frac{\partial f}{Q_3 \partial q_3} \quad \Leftrightarrow \quad g_i = \frac{\partial f}{Q_i \partial q_i} \tag{52}$$

$$\nabla\cdot\mathbf{g}=\nabla\cdot\nabla f=:\Delta f=\frac{1}{Q_1Q_2Q_3}\left[\frac{\partial\left(\frac{Q_2Q_3}{Q_1}\frac{\partial f}{\partial q_1}\right)}{\partial q_1}+\frac{\partial\left(\frac{Q_1Q_3}{Q_2}\frac{\partial f}{\partial q_2}\right)}{\partial q_2}+\frac{\partial\left(\frac{Q_1Q_2}{Q_3}\frac{\partial f}{\partial q_3}\right)}{\partial q_3}\right] \tag{53}$$

We can rewrite it for convenience in the following way:

$$\begin{aligned}\Delta f &= \frac{1}{Q_1Q_2Q_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{Q_2Q_3}{Q_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{Q_1Q_3}{Q_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{Q_1Q_2}{Q_3}\frac{\partial f}{\partial q_3}\right)\right\} && (54)\\ &= \frac{1}{Q_1Q_2Q_3}\left[\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(\frac{Q_1Q_2Q_3}{Q_i^2}\frac{\partial f}{\partial q_i}\right)\right] && (55)\\ &= \frac{1}{Q_1Q_2Q_3}\left[\sum_{i=1}^{3}\left(\frac{\partial}{\partial q_i}\left(\frac{Q_1Q_2Q_3}{Q_i^2}\right)\frac{\partial f}{\partial q_i}+\frac{Q_1Q_2Q_3}{Q_i^2}\frac{\partial^2 f}{\partial q_i^2}\right)\right] && (56)\end{aligned}$$

The next operator we calculate is the **curl of a vector field g**:

$$\nabla\times\mathbf{g}=\nabla\times\left(\sum_{i=1}^{3}g_i\mathbf{e_i}\right)=\sum_{i=1}^{3}\left(g_i\nabla\times\mathbf{e_i}+\nabla g_i\times\mathbf{e_i}\right) \tag{57}$$

where we used the vector identity $\nabla\times(g\mathbf{A})=g\nabla\times\mathbf{A}+\nabla g\times\mathbf{A}$ (see appendix for proof).
Considering cyclic permutations of equation (48) for $\nabla\times\mathbf{e_1}$ (namely the ones we already used in equation (49)) the only thing left we need is $\nabla g_i\times\mathbf{e_i}$ in order to write down equation (57) explicitely. We use the equation for the gradient developed before (using two different methods), equation (37), and change the indices (here j is the running index, since i is reserved for the index of g in this calculation) to do the last mentioned . We also use the fact that $\mathbf{e_i}\times\mathbf{e_j}=0$ for $i=j$:

$$\nabla g_i\times\mathbf{e_i}=\left(\sum_{j=1}^{3}\mathbf{e_j}\frac{\partial g_i}{Q_j\partial q_j}\right)\times\mathbf{e_i} \tag{58}$$

Hence we finally get our searched for expression:

$$\begin{aligned}\nabla\times\mathbf{g} &= \sum_{i=1}^{3}g_i\nabla\times\mathbf{e_i}+\sum_{i=1}^{3}\nabla g_i\times\mathbf{e_i}\\ &= g_1\left(\frac{\mathbf{e_2}}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-\frac{\mathbf{e_3}}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}\right)+g_2\left(\frac{\mathbf{e_3}}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}-\frac{\mathbf{e_1}}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}\right)+\ldots\\ &\quad g_3\left(\frac{\mathbf{e_1}}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}-\frac{\mathbf{e_2}}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}\right)+\ldots\\ &\quad \left(\frac{\partial g_1}{Q_2\partial q_2}\underbrace{\mathbf{e_2}\times\mathbf{e_1}}_{-\mathbf{e_3}}+\frac{\partial g_1}{Q_3\partial q_3}\underbrace{\mathbf{e_3}\times\mathbf{e_1}}_{\mathbf{e_2}}\right)+\left(\frac{\partial g_2}{Q_1\partial q_1}\underbrace{\mathbf{e_1}\times\mathbf{e_2}}_{\mathbf{e_3}}+\frac{\partial g_2}{Q_3\partial q_3}\underbrace{\mathbf{e_3}\times\mathbf{e_2}}_{-\mathbf{e_1}}\right)+\ldots\\ &\quad \left(\frac{\partial g_3}{Q_1\partial q_1}\underbrace{\mathbf{e_1}\times\mathbf{e_3}}_{-\mathbf{e_2}}+\frac{\partial g_3}{Q_2\partial q_2}\underbrace{\mathbf{e_2}\times\mathbf{e_3}}_{\mathbf{e_1}}\right)\\ &= \mathbf{e_1}\left(\frac{g_3}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}-\frac{g_2}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}+\frac{\partial g_3}{Q_2\partial q_2}-\frac{\partial g_2}{Q_3\partial q_3}\right)+\ldots\\ &\quad \mathbf{e_2}\left(\frac{g_1}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-\frac{g_3}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+\frac{\partial g_1}{Q_3\partial q_3}-\frac{\partial g_3}{Q_1\partial q_1}\right)+\ldots\\ &\quad \mathbf{e_3}\left(\frac{g_2}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}-\frac{g_1}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}+\frac{\partial g_2}{Q_1\partial q_1}-\frac{\partial g_1}{Q_2\partial q_2}\right)\\ &= \frac{\mathbf{e_1}}{Q_2Q_3}\left(\frac{\partial(Q_3g_3)}{\partial q_2}-\frac{\partial(Q_2g_2)}{\partial q_3}\right)+\frac{\mathbf{e_2}}{Q_1Q_3}\left(\frac{\partial(Q_1g_1)}{\partial q_3}-\frac{\partial(Q_3g_3)}{\partial q_1}\right)+\frac{\mathbf{e_3}}{Q_1Q_2}\left(\frac{\partial(Q_2g_2)}{\partial q_1}-\frac{\partial(Q_1g_1)}{\partial q_2}\right)\\ &= \sum_{i=1}^{3}\frac{\mathbf{e_i}Q_i}{Q_1Q_2Q_3}\sum_{j,k=1}^{3}\epsilon_{ijk}\frac{\partial}{\partial q_j}(Q_kg_k)=\sum_{i,j,k=1}^{3}\mathbf{e_i}\frac{\epsilon_{ijk}Q_i}{Q_1Q_2Q_3}\frac{\partial(Q_kg_k)}{\partial q_j} && (59)\end{aligned}$$

In the last short formula we realized that the terms belonging to $\mathbf{e_i}$ represented the i-th coordinate of a cross product of a formal vector with j-th component $\frac{\partial}{\partial q_j}$ and a vector with k-th component being $Q_k g_k$. It is Important to realize, that the unsymmetric identities we derive rely heavily on the fact, that $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ form a **righthanded system**! Thus, one should be careful when identifying the unit vectors in a curvilinear orthogonal coordinate system with the $\mathbf{e_i}$'s in the right order and also to identify the corresponding coordinates of the vectors in this space with the g_i's, or, later on, the A_i's and B_i's.
The last operator will be the **Laplacian of a vector field**, say **g**, which is a vector *defined* as:

$$\nabla^2 \mathbf{g} \quad =: \quad \Delta \mathbf{g} := (\nabla \cdot \nabla)\mathbf{g} = \nabla(\nabla \cdot \mathbf{g}) - \nabla \times (\nabla \times \mathbf{g}) \tag{60}$$

Where we will use equations (37), (51) and (59) we derived before for the gradient, divergence and the curl of a vector and the vector identity $\nabla \times (\nabla \times \mathbf{g}) = \nabla(\nabla \cdot \mathbf{g}) - (\nabla \cdot \nabla)\mathbf{g}$. Sometimes the symbol □ is used to distinguish the vector Laplacian from the scalar Laplacian. Expression (60) is somewhat nasty to write down and won't get too much easier writing it this way:

$$\begin{aligned}
&= \sum_{i=1}^{3} \mathbf{e_i} \left\{ \frac{\partial \left[\frac{1}{Q_1 Q_2 Q_3} \sum_{k=1}^{3} \frac{\partial}{\partial q_k} \left(Q_1 Q_2 Q_3 \frac{g_k}{Q_k} \right) \right]}{Q_i \partial q_i} - \sum_{j,k=1}^{3} \frac{\epsilon_{ijk} Q_i}{Q_1 Q_2 Q_3} \frac{\partial \left(Q_k \left[\sum_{l,m=1}^{3} \frac{\epsilon_{klm} Q_k}{Q_1 Q_2 Q_3} \frac{\partial (Q_m g_m)}{\partial q_l} \right] \right)}{\partial q_j} \right\} \\
&= \mathbf{e_1} \left(-\frac{\partial \left[\frac{Q_3}{Q_1 Q_2} \left(\frac{\partial (Q_2 g_2)}{\partial q_1} - \frac{\partial (Q_1 g_1)}{\partial q_2} \right) \right]}{Q_2 Q_3 \partial q_2} + \frac{\partial \left[\frac{Q_2}{Q_1 Q_3} \left(\frac{\partial (Q_1 g_1)}{\partial q_3} - \frac{\partial (Q_3 g_3)}{\partial q_1} \right) \right]}{Q_2 Q_3 \partial q_3} + \frac{\partial \left[\frac{1}{Q_1 Q_2 Q_3} \sum_{k=1}^{3} \frac{\partial}{\partial q_k} \left(Q_1 Q_2 Q_3 \frac{g_k}{Q_k} \right) \right]}{Q_1 \partial q_1} \right) + \ldots \\
&\quad \mathbf{e_2} \left(-\frac{\partial \left[\frac{Q_1}{Q_2 Q_3} \left(\frac{\partial (Q_3 g_3)}{\partial q_2} - \frac{\partial (Q_2 g_2)}{\partial q_3} \right) \right]}{Q_1 Q_3 \partial q_3} + \frac{\partial \left[\frac{Q_3}{Q_1 Q_2} \left(\frac{\partial (Q_2 g_2)}{\partial q_1} - \frac{\partial (Q_1 g_1)}{\partial q_2} \right) \right]}{Q_1 Q_3 \partial q_1} + \frac{\partial \left[\frac{1}{Q_1 Q_2 Q_3} \sum_{k=1}^{3} \frac{\partial}{\partial q_k} \left(Q_1 Q_2 Q_3 \frac{g_k}{Q_k} \right) \right]}{Q_2 \partial q_2} \right) + \ldots \\
&\quad \mathbf{e_3} \left(-\frac{\partial \left[\frac{Q_2}{Q_1 Q_3} \left(\frac{\partial (Q_1 g_1)}{\partial q_3} - \frac{\partial (Q_3 g_3)}{\partial q_1} \right] \right)}{Q_1 Q_2 \partial q_1} + \frac{\partial \left[\frac{Q_1}{Q_2 Q_3} \left(\frac{\partial (Q_3 g_3)}{\partial q_2} - \frac{\partial (Q_2 g_2)}{\partial q_3} \right) \right]}{Q_1 Q_2 \partial q_2} + \frac{\partial \left[\frac{1}{Q_1 Q_2 Q_3} \sum_{k=1}^{3} \frac{\partial}{\partial q_k} \left(Q_1 Q_2 Q_3 \frac{g_k}{Q_k} \right) \right]}{Q_3 \partial q_3} \right)
\end{aligned} \tag{61}$$

Using equation (37) without f, which is, so to say, the definition for the ∇ operator in curvilinear coordinates and $\mathbf{e_i} \cdot \mathbf{e_j} = \delta_{ij}$ one gets the so-called **convective operator** acting on a **scalar function** f:

$$\begin{aligned}
(\mathbf{g} \cdot \nabla) f &= \left[(g_1 \mathbf{e_1} + g_2 \mathbf{e_2} + g_3 \mathbf{e_3}) \left(\mathbf{e_1} \frac{\partial}{Q_1 \partial q_1} + \mathbf{e_2} \frac{\partial}{Q_2 \partial q_2} + \mathbf{e_3} \frac{\partial}{Q_3 \partial q_3} \right) \right] f \\
&= \frac{g_1}{Q_1} \frac{\partial f}{\partial q_1} + \frac{g_2}{Q_2} \frac{\partial f}{\partial q_2} + \frac{g_3}{Q_3} \frac{\partial f}{\partial q_3} = \sum_{i=1}^{3} \frac{g_i}{Q_i} \frac{\partial f}{\partial q_i}
\end{aligned} \tag{62}$$

We can get an explicit formula for the **convective operator** acting on a **vector field A** using the following two vector identities (adding eqns. (63)+(64), proofs for both given in the appendix):

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{63}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{64}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) + \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + 2(\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{65}$$

Hence we get a usable representation:

$$(\mathbf{B} \cdot \nabla)\mathbf{A} = \frac{1}{2} [\nabla \times (\mathbf{A} \times \mathbf{B}) + \nabla(\mathbf{A} \cdot \mathbf{B}) - \mathbf{A}(\nabla \cdot \mathbf{B}) + \mathbf{B}(\nabla \cdot \mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A})] \tag{66}$$

In this expression only known and derived expressions like curl of a vector, divergence of a vector and the gradient of a scalar, $A_i B_i$, are present and could therefore be expressed in a general way. However, it is an insanely lengthy equation for which an easier form will be derived using a different approach, basically by observation. Anyway, finally, just for completeness sake of this script I will write down the formula for the convective operator acting on a vector field **A** in curvilinear orthogonal coordinates:

$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \sum_{i,j,k=1}^{3}\mathbf{e_i}\frac{\epsilon_{ijk}Q_i}{2Q_1Q_2Q_3}\frac{\partial\left(Q_k\epsilon_{klm}A_lB_m\right)}{\partial q_j}+\sum_{i=1}^{3}\mathbf{e_i}\frac{\partial\left(A_jB_j\right)}{2Q_i\partial q_i}-\ldots\\
&\sum_{i=1}^{3}\mathbf{e_i}A_i\frac{1}{2Q_1Q_2Q_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(Q_1Q_2Q_3\frac{B_i}{Q_i}\right)+\sum_{i=1}^{3}\mathbf{e_i}B_i\frac{1}{2Q_1Q_2Q_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(Q_1Q_2Q_3\frac{A_i}{Q_i}\right)-\ldots\\
&\sum_{g=1}^{3}\mathbf{e_g}\epsilon_{ghi}A_h\sum_{j,k=1}^{3}\frac{\epsilon_{ijk}Q_i}{2Q_1Q_2Q_3}\frac{\partial\left(Q_kB_k\right)}{\partial q_j}-\sum_{g=1}^{3}\mathbf{e_g}\epsilon_{ghi}B_h\sum_{j,k=1}^{3}\frac{\epsilon_{ijk}Q_i}{2Q_1Q_2Q_3}\frac{\partial\left(Q_kA_k\right)}{\partial q_j}\\
&= \sum_{i=1}^{3}\mathbf{e_i}\left(\sum_{j,k=1}^{3}\frac{\epsilon_{ijk}Q_i}{2Q_1Q_2Q_3}\frac{\partial\left(Q_k\epsilon_{klm}A_lB_m\right)}{\partial q_j}+\frac{\partial\left(A_pB_p\right)}{2Q_i\partial q_i}\right)-\ldots\\
&\sum_{i=1}^{3}\mathbf{e_i}\left(\frac{A_i}{2Q_1Q_2Q_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(Q_1Q_2Q_3\frac{B_i}{Q_i}\right)+\frac{B_i}{2Q_1Q_2Q_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(Q_1Q_2Q_3\frac{A_i}{Q_i}\right)\right)-\ldots\\
&\sum_{i=1}^{3}\mathbf{e_i}\left(\epsilon_{ijk}A_j\sum_{l,m=1}^{3}\frac{\epsilon_{klm}Q_k}{2Q_1Q_2Q_3}\frac{\partial\left(Q_mB_m\right)}{\partial q_l}-\epsilon_{ijk}B_j\sum_{l,m=1}^{3}\frac{\epsilon_{klm}Q_k}{2Q_1Q_2Q_3}\frac{\partial\left(Q_mA_m\right)}{\partial q_l}\right)
\end{aligned}
\tag{67}
$$

For $\mathbf{e_1}$ we will write down the term explicitely, using eq. (66), and later on consider cyclic permutations to get the results for the other 3 coordinates and find the general formula.

$$
\begin{aligned}
&(2(\mathbf{B}\cdot\nabla)\mathbf{A})_1\\
&= \frac{A_1B_2-A_2B_1}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}-\frac{A_3B_1-A_1B_3}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}+\frac{\partial\left(A_1B_2-A_2B_1\right)}{Q_2\partial q_2}-\frac{\partial\left(A_3B_1-A_1B_3\right)}{Q_3\partial q_3}-\ldots\\
&A_2\left(\frac{B_2}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}-\frac{B_1}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}+\frac{\partial B_2}{Q_1\partial q_1}-\frac{\partial B_1}{Q_2\partial q_2}\right)+A_3\left(\frac{B_1}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-\frac{B_3}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+\frac{\partial B_1}{Q_3\partial q_3}-\frac{\partial B_3}{Q_1\partial q_1}\right)-\ldots\\
&B_2\left(\frac{A_2}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}-\frac{A_1}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}+\frac{\partial A_2}{Q_1\partial q_1}-\frac{\partial A_1}{Q_2\partial q_2}\right)+B_3\left(\frac{A_1}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-\frac{A_3}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+\frac{\partial A_1}{Q_3\partial q_3}-\frac{\partial A_3}{Q_1\partial q_1}\right)+\ldots\\
&\frac{\partial\left(A_1B_1+A_2B_2+A_3B_3\right)}{Q_1\partial q_1}-\frac{A_1}{Q_1Q_2Q_3}\left(\frac{\partial}{\partial q_1}(Q_2Q_3B_1)+\frac{\partial}{\partial q_2}(Q_1Q_3B_2)+\frac{\partial}{\partial q_3}(Q_1Q_2B_3)\right)+\ldots\\
&\frac{B_1}{Q_1Q_2Q_3}\left(\frac{\partial}{\partial q_1}(Q_2Q_3A_1)+\frac{\partial}{\partial q_2}(Q_1Q_3A_2)+\frac{\partial}{\partial q_3}(Q_1Q_2A_3)\right)\\
&= \frac{A_1B_2}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}-\frac{A_2B_1}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}-\frac{A_3B_1}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}+\frac{A_1B_3}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}+\frac{A_1}{Q_2}\frac{\partial B_2}{\partial q_2}+\frac{B_2}{Q_2}\frac{\partial A_1}{\partial q_2}-\frac{A_2}{Q_2}\frac{\partial B_1}{\partial q_2}-\frac{B_1}{Q_2}\frac{\partial A_2}{\partial q_2}-\ldots\\
&\frac{A_3}{Q_3}\frac{\partial B_1}{\partial q_3}-\frac{B_1}{Q_3}\frac{\partial A_3}{\partial q_3}+\frac{A_1}{Q_3}\frac{\partial B_3}{\partial q_3}+\frac{B_3}{Q_3}\frac{\partial A_1}{\partial q_3}-\ldots\\
&\frac{A_2B_2}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}+\frac{A_2B_1}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}-\frac{A_2}{Q_1}\frac{\partial B_2}{\partial q_1}+\frac{A_2}{Q_2}\frac{\partial B_1}{\partial q_2}+\frac{A_3B_1}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-\frac{A_3B_3}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+\frac{A_3}{Q_3}\frac{\partial B_1}{\partial q_3}-\frac{A_3}{Q_1}\frac{\partial B_3}{\partial q_1}-\ldots\\
&\frac{B_2A_2}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}+\frac{B_2A_1}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}-\frac{B_2}{Q_1}\frac{\partial A_2}{\partial q_1}+\frac{B_2}{Q_2}\frac{\partial A_1}{\partial q_2}+\frac{B_3A_1}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-\frac{B_3A_3}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+\frac{B_3}{Q_3}\frac{\partial A_1}{\partial q_3}-\frac{B_3}{Q_1}\frac{\partial A_3}{\partial q_1}+\ldots\\
&\frac{A_1}{Q_1}\frac{\partial B_1}{\partial q_1}+\frac{B_1}{Q_1}\frac{\partial A_1}{\partial q_1}+\frac{A_2}{Q_1}\frac{\partial B_2}{\partial q_1}+\frac{B_2}{Q_1}\frac{\partial A_2}{\partial q_1}+\frac{A_3}{Q_1}\frac{\partial B_3}{\partial q_1}+\frac{B_3}{Q_1}\frac{\partial A_3}{\partial q_1}-\ldots\\
&\frac{A_1}{Q_1}\frac{\partial B_1}{\partial q_1}-\frac{A_1}{Q_2}\frac{\partial B_2}{\partial q_2}-\frac{A_1}{Q_3}\frac{\partial B_3}{\partial q_3}-\frac{A_1B_1}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}-\frac{A_1B_1}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}-\frac{A_1B_2}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}-\frac{A_1B_2}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}-\frac{A_1B_3}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}-\frac{A_1B_3}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}+\ldots\\
&\frac{B_1}{Q_1}\frac{\partial A_1}{\partial q_1}+\frac{B_1}{Q_2}\frac{\partial A_2}{\partial q_2}+\frac{B_1}{Q_3}\frac{\partial A_3}{\partial q_3}+\frac{B_1A_1}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+\frac{B_1A_1}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}+\frac{B_1A_2}{Q_2Q_3}\frac{\partial Q_3}{\partial q_2}+\frac{B_1A_2}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}+\frac{B_1A_3}{Q_2Q_3}\frac{\partial Q_2}{\partial q_3}+\frac{B_1A_3}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}\\
&= 2\frac{B_2}{Q_2}\frac{\partial A_1}{\partial q_2}+2\frac{B_3}{Q_3}\frac{\partial A_1}{\partial q_3}-2\frac{A_2B_2}{Q_1Q_2}\frac{\partial Q_2}{\partial q_1}+2\frac{A_2B_1}{Q_1Q_2}\frac{\partial Q_1}{\partial q_2}+2\frac{A_3B_1}{Q_1Q_3}\frac{\partial Q_1}{\partial q_3}-2\frac{A_3B_3}{Q_1Q_3}\frac{\partial Q_3}{\partial q_1}+2\frac{B_1}{Q_1}\frac{\partial A_1}{\partial q_1}
\end{aligned}
\tag{68}
$$

Where in the last line expressions are noted in the order of appereance of non-vanishing twice appearing terms. (You may print out the page and cancel the terms by yourself to check the result)

$$((\mathbf{B}\cdot\nabla)\mathbf{A})_1 = \sum_{k=1}^{3}\frac{B_k}{Q_k}\frac{\partial A_1}{\partial q_k}+\frac{A_2}{Q_1Q_2}\left(B_1\frac{\partial Q_1}{\partial q_2}-B_2\frac{\partial Q_2}{\partial q_1}\right)+\frac{A_3}{Q_1Q_3}\left(B_1\frac{\partial Q_1}{\partial q_3}-B_3\frac{\partial Q_3}{\partial q_1}\right) \tag{69}$$

$$((\mathbf{B}\cdot\nabla)\mathbf{A})_2 = \sum_{k=1}^{3}\frac{B_k}{Q_k}\frac{\partial A_2}{\partial q_k}+\frac{A_3}{Q_2Q_3}\left(B_2\frac{\partial Q_2}{\partial q_3}-B_3\frac{\partial Q_3}{\partial q_2}\right)+\frac{A_1}{Q_1Q_2}\left(B_2\frac{\partial Q_2}{\partial q_1}-B_1\frac{\partial Q_1}{\partial q_2}\right) \tag{70}$$

$$((\mathbf{B}\cdot\nabla)\mathbf{A})_3 = \sum_{k=1}^{3}\frac{B_k}{Q_k}\frac{\partial A_3}{\partial q_k}+\frac{A_1}{Q_1Q_3}\left(B_3\frac{\partial Q_3}{\partial q_1}-B_1\frac{\partial Q_1}{\partial q_3}\right)+\frac{A_2}{Q_2Q_3}\left(B_3\frac{\partial Q_3}{\partial q_2}-B_2\frac{\partial Q_2}{\partial q_3}\right) \tag{71}$$

Where cyclic permutations of 123 were used to get the formulas for the second and third coordinate. Summarizing, we can write down the general formula for the convective operator in curvilinear orthogonal coordinates:

$$(\mathbf{B}\cdot\nabla)\mathbf{A}=\sum_{i=1}^{3}\mathbf{e_i}\left\{\sum_{k=1}^{3}\frac{B_k}{Q_k}\frac{\partial A_i}{\partial q_k}+\sum_{j=1}^{3}\frac{A_j}{Q_iQ_j}\left(B_i\frac{\partial Q_i}{\partial q_j}-B_j\frac{\partial Q_j}{\partial q_i}\right)\right\} \tag{72}$$

For a **curvilinear orthogonal *righthanded*** (only important for the last 3) **coordinate system** we have

$$x = x(q_1,q_2,q_3);\quad y = y(q_1,q_2,q_3);\quad z = z(q_1,q_2,q_3);\quad \mathbf{e_i} = \frac{1}{Q_i}\frac{\partial \mathbf{r}(q_1,q_2,q_3)}{\partial q_i}$$

$$Q_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2};\qquad \mathrm{d}V = \prod_{i=1}^{3}Q_i\mathrm{d}q_i$$

$$\mathrm{d}s = \sqrt{\sum_{i=1}^{3}Q_i^2\mathrm{d}q_i^2};\qquad \mathrm{d}\mathbf{r} = \sum_{i=1}^{3}Q_i\mathrm{d}q_i\mathbf{e_i}$$

$$\nabla f = \sum_{i=1}^{3}\mathbf{e_i}\left\{\frac{\partial f}{Q_i\partial q_i}\right\};\qquad \left|\nabla f\right|^2 = \sum_{i=1}^{3}\frac{1}{Q_i^2}\left(\frac{\partial f}{\partial q_i}\right)^2$$

$$\Delta f = \frac{1}{Q_1Q_2Q_3}\left[\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(\frac{Q_1Q_2Q_3}{Q_i^2}\frac{\partial f}{\partial q_i}\right)\right];\qquad \nabla\cdot\mathbf{A} = \frac{1}{Q_1Q_2Q_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(Q_1Q_2Q_3\frac{A_i}{Q_i}\right)$$

$$(\mathbf{A}\cdot\nabla)f = \sum_{i=1}^{3}\frac{A_i}{Q_i}\frac{\partial f}{\partial q_i};\qquad v^2 = \sum_{i=1}^{3}Q_i^2\dot{q_i}^2$$

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = \sum_{i=1}^{3}\mathbf{e_i}\left\{\sum_{k=1}^{3}\left[\frac{B_k}{Q_k}\frac{\partial A_i}{\partial q_k}+\frac{A_k}{Q_iQ_k}\left(B_i\frac{\partial Q_i}{\partial q_k}-B_k\frac{\partial Q_k}{\partial q_i}\right)\right]\right\}$$

$$\nabla\times\mathbf{A} = \sum_{i=1}^{3}\mathbf{e_i}\left\{\frac{Q_i}{Q_1Q_2Q_3}\sum_{j,k=1}^{3}\epsilon_{ijk}\frac{\partial}{\partial q_j}(Q_kA_k)\right\}$$

$$\Delta\mathbf{A} = \sum_{i=1}^{3}\mathbf{e_i}\left\{\frac{\partial}{Q_i\partial q_i}\left(\frac{1}{Q_1Q_2Q_3}\sum_{k=1}^{3}\frac{\partial}{\partial q_k}\left(Q_1Q_2Q_3\frac{A_k}{Q_k}\right)\right)-\sum_{j,k=1}^{3}\frac{Q_i\epsilon_{ijk}}{Q_1Q_2Q_3}\frac{\partial}{\partial q_j}\left(\sum_{l,m=1}^{3}\frac{\epsilon_{klm}Q_k^2}{Q_1Q_2Q_3}\frac{\partial(Q_mA_m)}{\partial q_l}\right)\right\}$$

4 Examples using the General Expressions

These examples demonstarte the ease of use of the derived equations, first for **spherical coordinates**:

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi; \quad z = r\cos\theta; \quad \left\{\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi, \right\} \tag{73}$$

$$Q_r^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1 \qquad \Rightarrow Q_r = 1 \tag{74}$$

$$Q_\theta^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + \sin^2\theta = r^2 \qquad \Rightarrow Q_\theta = r \tag{75}$$

$$Q_\phi^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta \qquad \Rightarrow Q_\phi = r\sin\theta \tag{76}$$

$$\Delta f = \frac{1}{Q_r Q_\theta Q_\phi}\left\{\frac{\partial}{\partial r}\left(\frac{Q_\theta Q_\phi}{Q_r}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{Q_r Q_\phi}{Q_\theta}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{Q_r Q_\theta}{Q_\phi}\frac{\partial f}{\partial \phi}\right)\right\} \tag{77}$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta}{1}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{78}$$

$$\nabla f = \mathbf{e_r}\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \tag{79}$$

$$\nabla\times\mathbf{g} = \frac{\mathbf{e_r}}{Q_\theta Q_\phi}\left(\frac{\partial\left(Q_\phi g_\phi\right)}{\partial\theta} - \frac{\partial\left(Q_\theta g_\theta\right)}{\partial\phi}\right) + \frac{\mathbf{e}_\theta}{Q_r Q_\phi}\left(\frac{\partial\left(Q_r g_r\right)}{\partial\phi} - \frac{\partial\left(Q_\phi g_\phi\right)}{\partial r}\right) + \frac{\mathbf{e}_\phi}{Q_r Q_\theta}\left(\frac{\partial\left(Q_\theta g_\theta\right)}{\partial r} - \frac{\partial\left(Q_r g_r\right)}{\partial\theta}\right)$$

$$= \frac{\mathbf{e_r}}{r^2\sin\theta}\left(\frac{\partial\left(r\sin\theta g_\phi\right)}{\partial\theta} - \frac{\partial\left(r g_\theta\right)}{\partial\phi}\right) + \frac{\mathbf{e}_\theta}{r\sin\theta}\left(\frac{\partial\left(g_r\right)}{\partial\phi} - \frac{\partial\left(r\sin\theta g_\phi\right)}{\partial r}\right) + \frac{\mathbf{e}_\phi}{r}\left(\frac{\partial\left(r g_\theta\right)}{\partial r} - \frac{\partial\left(g_r\right)}{\partial\theta}\right)$$

$$= \frac{\mathbf{e_r}}{r\sin\theta}\left(\frac{\partial\left(\sin\theta g_\phi\right)}{\partial\theta} - \frac{\partial g_\theta}{\partial\phi}\right) + \frac{\mathbf{e}_\theta}{r}\left(\frac{\partial g_r}{\sin\theta\partial\phi} - \frac{\partial\left(r g_\phi\right)}{\partial r}\right) + \frac{\mathbf{e}_\phi}{r}\left(\frac{\partial\left(r g_\theta\right)}{\partial r} - \frac{\partial g_r}{\partial\theta}\right) \tag{80}$$

$$\nabla\cdot\mathbf{g} = \frac{1}{Q_r Q_\theta Q_\phi}\left[\frac{\partial}{\partial r}\left(Q_\theta Q_\phi g_r\right) + \frac{\partial}{\partial\theta}\left(Q_r Q_\phi g_\theta\right) + \frac{\partial}{\partial\phi}\left(Q_r Q_\theta g_\phi\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta g_r\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta g_\theta\right) + \frac{\partial}{\partial\phi}\left(r g_\phi\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta g_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(g_\phi\right) \tag{81}$$

$$\Delta\mathbf{g} = -\nabla\times\left(\nabla\times\mathbf{g}\right) + \nabla\left(\nabla\cdot\mathbf{g}\right)$$

$$= -\frac{\mathbf{e_r}}{r\sin\theta}\left(\frac{\partial\left[\frac{\sin\theta}{r}\left(\frac{\partial\left(r g_\theta\right)}{\partial r} - \frac{\partial g_r}{\partial\theta}\right)\right]}{\partial\theta} - \frac{\partial\left[\frac{1}{r}\left(\frac{\partial g_r}{\sin\theta\partial\phi} - \frac{\partial\left(r g_\phi\right)}{\partial r}\right)\right]}{\partial\phi}\right) - \ldots$$

$$\frac{\mathbf{e}_\theta}{r}\left(\frac{\partial\left[\frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta g_\phi\right)}{\partial\theta} - \frac{\partial g_\theta}{\partial\phi}\right)\right]}{\sin\theta\partial\phi} - \frac{\partial\left[\left(\frac{\partial\left(r g_\theta\right)}{\partial r} - \frac{\partial g_r}{\partial\theta}\right)\right]}{\partial r}\right) - \ldots$$

$$\frac{\mathbf{e}_\phi}{r}\left(\frac{\partial\left(\left[\left(\frac{\partial g_r}{\sin\theta\partial\phi} - \frac{\partial\left(r g_\phi\right)}{\partial r}\right)\right]\right)}{\partial r} - \frac{\partial\left[\frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta g_\phi\right)}{\partial\theta} - \frac{\partial g_\theta}{\partial\phi}\right)\right]}{\partial\theta}\right) + \ldots$$

$$\begin{aligned}
& \mathbf{e_r}\frac{\partial\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, g_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(g_\phi\right)\right]}{\partial r}+\ldots\\
& \mathbf{e}_\theta\frac{\partial\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, g_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(g_\phi\right)\right]}{r\partial\theta}+\ldots\\
& \mathbf{e}_\phi\frac{\partial\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, g_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(g_\phi\right)\right]}{r\sin\theta\partial\phi}\\
= & \mathbf{e_r}\left(\frac{1}{r}\frac{\partial^2(rg_r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 g_r}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 g_r}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial g_r}{\partial\theta}-\frac{2}{r^2}\frac{\partial g_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial g_\phi}{\partial\phi}-\frac{2g_r}{r^2}-\frac{2\cot\theta}{r^2}g_\theta\right)+\ldots\\
& \mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial^2(rg_\theta)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 g_\theta}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 g_\theta}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial g_\theta}{\partial\theta}-\frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial g_\phi}{\partial\phi}+\frac{2}{r^2}\partial g_r\partial\theta-\frac{g_\theta}{r^2\sin^2\theta}\right)+\ldots\\
& \mathbf{e}_\phi\left(\frac{1}{r}\frac{\partial^2(rg_\phi)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 g_\phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 g_\phi}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial g_\phi}{\partial\theta}+\frac{2}{r^2\sin\theta}\frac{\partial g_r}{\partial\phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial g_\theta}{\partial\phi}-\frac{g_\phi}{r^2\sin^2\theta}\right)\\
= & \mathbf{e_r}\left(\Delta g_r-\frac{2}{r^2}\frac{\partial g_\theta}{\partial\theta}-\frac{2g_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial g_\phi}{\partial\phi}\right)+\ldots\\
& \mathbf{e}_\theta\left(\Delta g_\theta+\frac{2}{r^2}\frac{\partial g_r}{\partial\theta}-\frac{g_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial g_\phi}{\partial\phi}\right)+\ldots\\
& \mathbf{e}_\phi\left(\Delta g_\phi-\frac{g_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial g_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial g_\theta}{\partial\phi}\right) \quad (82)
\end{aligned}$$

(83)

Using Scharz' theorem many derivatives dropped out and terms vanished. But still, it would take a lot of time to get the result stated in the last lines. (Forget the before mentioned *ease* for this one)

$$\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \sum_{i=r,\phi,\theta}\mathbf{e_i}\left\{\sum_{k=r,\phi,\theta}\frac{B_k}{Q_k}\frac{\partial A_i}{\partial q_k}+\sum_{j=r,\phi,\theta}\frac{A_j}{Q_iQ_j}\left(B_i\frac{\partial Q_i}{\partial q_j}-B_j\frac{\partial Q_j}{\partial q_i}\right)\right\}\\
&= \mathbf{e_r}\left(B_r\frac{\partial A_r}{\partial r}+\frac{B_\theta}{r}\frac{\partial A_r}{\partial\theta}+\frac{B_\phi}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{B_\theta A_\theta+B_\phi A_\phi}{r}\right)+\ldots\\
&\quad \mathbf{e}_\theta\left(B_r\frac{\partial A_\theta}{\partial r}+\frac{B_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{B_\phi}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}+\frac{B_\theta A_r}{r}-\frac{B_\phi A_\phi\cot\theta}{r}\right)+\ldots\\
&\quad \mathbf{e}_\phi\left(B_r\frac{\partial A_\phi}{\partial r}+\frac{B_\theta}{r}\frac{\partial A_\phi}{\partial\theta}+\frac{B_\phi}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{B_\phi A_r}{r}+\frac{B_\phi A_\theta\cot\theta}{r}\right) \quad (84)
\end{aligned}$$

(85)

For **cylindrical coordinates** we find the following identities.

$$z = z;\quad x = r\cos\phi;\quad y = r\sin\phi;\quad \left\{\mathbf{e_r},\mathbf{e}_\phi,\mathbf{e_z},\right\} \quad (86)$$

$$Q_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2} \quad (87)$$

$$Q_r^2 = 0^2+r^2\cos^2\phi+r^2\sin^2\phi=1 \qquad \Rightarrow Q_r=1 \quad (88)$$

$$Q_\phi^2 = 0^2+r^2\sin^2\phi+r^2\cos^2\phi=r^2 \qquad \Rightarrow Q_\phi=r \quad (89)$$

$$Q_z^2 = 1^2+0^2+0^2=1 \qquad \Rightarrow Q_z=1 \quad (90)$$

$$\begin{aligned}
\Delta f &= \frac{1}{Q_rQ_zQ_\phi}\left\{\frac{\partial}{\partial r}\left(\frac{Q_\phi Q_z}{Q_r}\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{Q_rQ_z}{Q_\phi}\frac{\partial f}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\frac{Q_rQ_\phi}{Q_z}\frac{\partial f}{\partial z}\right)\right\}\\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial f}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2} \quad (91)
\end{aligned}$$

$$\nabla f = \mathbf{e_r}\frac{\partial f}{\partial r}+\mathbf{e}_\phi\frac{1}{r}\frac{\partial f}{\partial \phi}+\mathbf{e_z}\frac{\partial f}{\partial z} \tag{92}$$

$$\begin{aligned}
\nabla\times\mathbf{g} &= \frac{\mathbf{e_r}}{Q_\phi Q_z}\left(\frac{\partial\left(Q_z g_z\right)}{\partial\phi}-\frac{\partial\left(Q_\phi g_\phi\right)}{\partial z}\right)+\frac{\mathbf{e}_\phi}{Q_r Q_z}\left(\frac{\partial\left(Q_r g_r\right)}{\partial z}-\frac{\partial\left(Q_z g_z\right)}{\partial r}\right)+\frac{\mathbf{e_z}}{Q_r Q_\phi}\left(\frac{\partial\left(Q_\phi g_\phi\right)}{\partial r}-\frac{\partial\left(Q_r g_r\right)}{\partial\phi}\right)\\
&= \frac{\mathbf{e_r}}{r}\left(\frac{\partial\left(g_z\right)}{\partial\phi}-\frac{\partial\left(r g_\phi\right)}{\partial z}\right)+\mathbf{e}_\phi\left(\frac{\partial\left(g_r\right)}{\partial z}-\frac{\partial\left(g_z\right)}{\partial r}\right)+\frac{\mathbf{e_z}}{r}\left(\frac{\partial\left(r g_\phi\right)}{\partial r}-\frac{\partial\left(g_r\right)}{\partial\phi}\right)\\
&= \mathbf{e_r}\left(\frac{1}{r}\frac{\partial g_z}{\partial\phi}-\frac{\partial g_\phi}{\partial z}\right)+\mathbf{e}_\phi\left(\frac{\partial g_r}{\partial z}-\frac{\partial g_z}{\partial r}\right)+\mathbf{e_z}\left(\frac{1}{r}\frac{\partial\left(r g_\phi\right)}{\partial r}-\frac{1}{r}\frac{\partial g_r}{\partial\phi}\right)
\end{aligned} \tag{93}$$

$$\begin{aligned}
\nabla\cdot\mathbf{g} &= \frac{1}{Q_r Q_\phi Q_\phi}\left[\frac{\partial}{\partial r}\left(Q_\phi Q_z g_r\right)+\frac{\partial}{\partial\phi}\left(Q_r Q_z g_\phi\right)+\frac{\partial}{\partial z}\left(Q_r Q_\phi g_z\right)\right]\\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r g_r\right)+\frac{\partial}{\partial\phi}\left(g_\phi\right)+\frac{\partial}{\partial z}\left(r g_z\right)\right]\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r g_r\right)+\frac{1}{r}\frac{\partial g_\phi}{\partial\phi}+\frac{\partial g_z}{\partial z}
\end{aligned} \tag{94}$$

$$\begin{aligned}
\Delta\mathbf{g} &= -\nabla\times\left(\nabla\times\mathbf{g}\right)+\nabla\left(\nabla\cdot\mathbf{g}\right)\\
&= -\mathbf{e_r}\left(\frac{1}{r}\frac{\partial\left(\frac{1}{r}\frac{\partial(r g_\phi)}{\partial r}-\frac{1}{r}\frac{\partial g_r}{\partial\phi}\right)}{\partial\phi}-\frac{\partial\left(\frac{\partial g_r}{\partial z}-\frac{\partial g_z}{\partial r}\right)}{\partial z}\right)+\mathbf{e_r}\frac{\partial\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r g_r\right)+\frac{1}{r}\frac{\partial g_\phi}{\partial\phi}+\frac{\partial g_z}{\partial z}\right)}{\partial r}-\ldots\\
&\quad \mathbf{e}_\phi\left(\frac{\partial\left(\frac{1}{r}\frac{\partial g_z}{\partial\phi}-\frac{\partial g_\phi}{\partial z}\right)}{\partial z}-\frac{\partial\left(\frac{1}{r}\frac{\partial(r g_\phi)}{\partial r}-\frac{1}{r}\frac{\partial g_r}{\partial\phi}\right)}{\partial r}\right)+\mathbf{e}_\phi\frac{1}{r}\frac{\partial\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r g_r\right)+\frac{1}{r}\frac{\partial g_\phi}{\partial\phi}+\frac{\partial g_z}{\partial z}\right)}{\partial\phi}-\ldots\\
&\quad \mathbf{e_z}\left(\frac{1}{r}\frac{\partial\left(r\left(\frac{\partial g_r}{\partial z}-\frac{\partial g_z}{\partial r}\right)\right)}{\partial r}-\frac{1}{r}\frac{\partial\left(\frac{1}{r}\frac{\partial g_z}{\partial\phi}-\frac{\partial g_\phi}{\partial z}\right)}{\partial\phi}\right)+\mathbf{e_z}\frac{\partial\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r g_r\right)+\frac{1}{r}\frac{\partial g_\phi}{\partial\phi}+\frac{\partial g_z}{\partial z}\right)}{\partial z}\\
&= \mathbf{e_r}\left(\frac{\partial^2 g_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 g_r}{\partial\phi^2}+\frac{\partial^2 g_r}{\partial z^2}+\frac{1}{r}\frac{\partial g_r}{\partial r}-\frac{2}{r^2}\frac{\partial g_\phi}{\partial\phi}-\frac{g_r}{r^2}\right)+\ldots\\
&\quad \mathbf{e}_\phi\left(\frac{\partial^2 g_\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 g_\phi}{\partial\phi^2}+\frac{\partial^2 g_\phi}{\partial z^2}+\frac{1}{r}\frac{\partial g_\phi}{\partial r}+\frac{2}{r^2}\frac{\partial g_r}{\partial\phi}-\frac{g_\phi}{r^2}\right)+\ldots\\
&\quad \mathbf{e_z}\left(\frac{\partial^2 g_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 g_z}{\partial\phi^2}+\frac{\partial^2 g_z}{\partial z^2}+\frac{1}{r}\frac{\partial g_z}{\partial r}\right)\\
&= \mathbf{e_r}\left(\Delta g_r-\frac{2}{r^2}\frac{\partial g_\phi}{\partial\phi}-\frac{g_r}{r^2}\right)+\mathbf{e}_\phi\left(\Delta g_\phi+\frac{2}{r^2}\frac{\partial g_r}{\partial\phi}-\frac{g_\phi}{r^2}\right)+\mathbf{e_z}\left(\Delta g_z\right)
\end{aligned} \tag{95}$$

$$\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \sum_{i=r,z,\phi}\mathbf{e_i}\left\{\sum_{k=r,z,\phi}\frac{B_k}{Q_k}\frac{\partial A_i}{\partial q_k}+\sum_{j=r,z,\phi}\frac{A_j}{Q_i Q_j}\left(B_i\frac{\partial Q_i}{\partial q_j}-B_j\frac{\partial Q_j}{\partial q_i}\right)\right\}\\
&= \mathbf{e_r}\left(B_r\frac{\partial A_r}{\partial r}+\frac{B_\phi}{r}\frac{\partial A_r}{\partial\phi}+B_z\frac{\partial A_r}{\partial z}-\frac{B_\phi A_\phi}{r}\right)+\ldots\\
&\quad \mathbf{e}_\phi\left(B_r\frac{\partial A_\phi}{\partial r}+\frac{B_\phi}{r}\frac{\partial A_\phi}{\partial\phi}+B_z\frac{\partial A_\phi}{\partial z}+\frac{B_\phi A_r}{r}\right)+\ldots\\
&\quad \mathbf{e_z}\left(B_r\frac{\partial A_z}{\partial r}+\frac{B_\phi}{r}\frac{\partial A_z}{\partial\phi}+B_z\frac{\partial A_z}{\partial z}\right)
\end{aligned} \tag{96}$$

An important thing to note is, that only in **cartesian coordinates**, the Laplacian of a vector field decomposes solemly to the Laplacian of each coordinate times the corresponding unit vector (using eqn. (62)):

$$\Delta\mathbf{g} = (\nabla\cdot\nabla)\mathbf{g} = -\nabla\times(\nabla\times\mathbf{g}) + \nabla(\nabla\cdot\mathbf{g}) = \mathbf{e_x}\Delta g_x + \mathbf{e_y}\Delta g_y + \mathbf{e_z}\Delta g_z \tag{97}$$

Since we have for the simple cartesian case:

$$\Delta\mathbf{g} = -\begin{pmatrix} \frac{\partial(\partial g_y/\partial x - \partial g_x/\partial y)}{\partial y} - \frac{\partial(\partial g_x/\partial z - \partial g_z/\partial x)}{\partial z} \\ \frac{\partial(\partial g_x/\partial z - \partial g_z/\partial x)}{\partial z} - \frac{\partial(\partial g_y/\partial x - \partial g_x/\partial y)}{\partial x} \\ \frac{\partial(\partial g_x/\partial z - \partial g_z/\partial x)}{\partial x} - \frac{\partial(\partial g_z/\partial y - \partial g_y/\partial z)}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2 g_x}{\partial x^2} + \frac{\partial^2 g_y}{\partial x\partial y} + \frac{\partial^2 g_z}{\partial x\partial z} \\ \frac{\partial^2 g_x}{\partial x\partial y} + \frac{\partial^2 g_y}{\partial y^2} + \frac{\partial^2 g_z}{\partial y\partial z} \\ \frac{\partial^2 g_x}{\partial x\partial z} + \frac{\partial^2 g_y}{\partial y\partial z} + \frac{\partial^2 g_z}{\partial z^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 g_x}{\partial x^2} + \frac{\partial^2 g_x}{\partial y^2} + \frac{\partial^2 g_x}{\partial z^2} \\ \frac{\partial^2 g_y}{\partial x^2} + \frac{\partial^2 g_y}{\partial y^2} + \frac{\partial^2 g_y}{\partial z^2} \\ \frac{\partial^2 g_z}{\partial x^2} + \frac{\partial^2 g_z}{\partial y^2} + \frac{\partial^2 g_z}{\partial z^2} \end{pmatrix}$$

Also notice the simplified version of the convective operator in the case of cartesian coordinates.

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = \sum_{i=x,y,z}\left\{\left(\sum_{j=x,y,z} B_j\frac{\partial}{\partial x_j}\right)A_i\right\}\mathbf{e_i} = \sum_{i=x,y,z}\left\{\sum_{j=x,y,z} B_j\frac{\partial A_i}{\partial x_j}\right\}\mathbf{e_i} = \begin{pmatrix} \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_x \\ \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_y \\ \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_z \end{pmatrix} \tag{98}$$

Looking back at page 6 we can find further interesting facts: For example we get for the differential ds of the **line-element**, using orthonormality again ($e_k\cdot e_l = \delta_{lk}$) and the expression we found for d**r**:

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = Q_1^2\mathrm{d}q_1^2 + Q_2^2\mathrm{d}q_2^2 + Q_3^2\mathrm{d}q_3^2 \quad\Rightarrow\quad \mathrm{d}s = \sqrt{\sum_{i=1}^{3} Q_i^2\mathrm{d}q_i^2} \tag{99}$$

The infitisimal **volume element** is obviously easily calculable using the parallelepipidial product of the scaled parts of the vector d**r**:

$$\mathrm{d}V = \left\|(Q_1\mathrm{d}q_1\mathbf{e_1})\cdot\left[(Q_2\mathrm{d}q_2\mathbf{e_2})\times(Q_3\mathrm{d}q_3\mathbf{e_3})\right]\right\| = Q_1Q_2Q_3\ \mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 = \prod_{i=1}^{3} Q_i\mathrm{d}q_i \tag{100}$$

Where we used $\|\mathbf{e_1}\cdot(\mathbf{e_2}\times\mathbf{e_3})\| = 1$. For spherical and cylindrical coordinates we get the well-known expressions which are also derivable via graphical observation or, more formally, from the Jacobian due to the general change-of-variables-theorem:

$$\mathrm{d}V_{\text{sph.c.}} = r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{101}$$

$$\mathrm{d}V_{\text{cyl.c.}} = r\ \mathrm{d}r\mathrm{d}z\mathrm{d}\phi \tag{102}$$

As another application in between we consider the **velocity** in curvilinear coordinates using again the derived expression for d**r** and differentiating it with respect to time.

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{\mathbf{r}} = \mathbf{v} = \sum_{i=1}^{3}\frac{\partial\mathbf{r}}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} = Q_1\mathbf{e_1}\frac{\mathrm{d}q_1}{\mathrm{d}t} + Q_2\mathbf{e_2}\frac{\mathrm{d}q_2}{\mathrm{d}t} + Q_3\mathbf{e_3}\frac{\mathrm{d}q_3}{\mathrm{d}t} = \sum_{i=1}^{3} Q_i\,\dot{q}_i\,\mathbf{e_i} \tag{103}$$

Hence we get for the absolute value of the velocity (acceleration is more involved!), which is important for expressing the kinetic energy $\frac{1}{2}mv^2$ in dependence on the curvilinear components of the velocity:

$$v^2 = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = Q_1^2\,\dot{q}_1^{\,2} + Q_2^2\,\dot{q}_2^{\,2} + Q_3^2\,\dot{q}_3^{\,2} \quad\Rightarrow\quad |\mathbf{v}| = \sqrt{\sum_{i=1}^{3} Q_i^2\,\dot{q}_i^{\,2}} \tag{104}$$

For spherical and cylindrical coordinates we get, by using common coordinate notation:

$$v^2_{\text{sph.c.}} = \dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 \tag{105}$$

$$v^2_{\text{cyl.c.}} = \dot{r}^2 + \dot{z}^2 + r^2\,\dot{\phi}^2 \tag{106}$$

5 Proofs for the ∇ Product Rules

Throughout this section we will make extensive use of the Einstein summation convention (summation over every repeated index), the Kronecker-Delta δ_{ij} (1 for equal indices, otherwise: 0) and the Levi-Civita-Symbol/Pseudo-Tensor (sometimes called the permutation symbol) ϵ (1 for even permiations, -1 for odd permutations, 0 for at least two equal indices). Definitions can be found in [5] for example.

$$\epsilon_{ijk} = \begin{cases} 1 & (ijk)=(123),(231),(312), \quad \text{even permutations} \\ -1 & (ijk)=(321),(213),(132), \quad \text{odd permutations} \\ 0 & (ijk)=(112),(221),\ldots, \quad \text{at least 2 equal indices} \end{cases} \quad ; \quad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{107}$$

$$\begin{aligned}
\epsilon_{ijk} &:= \mathbf{e_i}\cdot\left(\mathbf{e_j}\times\mathbf{e_k}\right) \qquad \text{for a right-handed coordinate system} \\
\mathbf{a}\times\mathbf{b} &= \sum_{i=1}^{3}\mathbf{e_i}\left(\sum_{j,k=1}^{3}\epsilon_{ijk}a_j b_k\right) = \mathbf{e_i}\epsilon_{ijk}a_j b_k \\
(\mathbf{a}\times\mathbf{b})_i &= \sum_{j,k=1}^{3}\epsilon_{ijk}a_j b_k = \epsilon_{ijk}a_j b_k \\
\mathbf{a}\cdot\mathbf{b} &= \sum_{i=1}^{3}a_i b_i = a_i b_i \\
\mathbf{a} &= \sum_{i=1}^{3}\mathbf{e_i}a_i = \mathbf{e_i}a_i \\
\epsilon_{ijk}\epsilon_{klm} &= \sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\
\epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} \quad = \quad -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}
\end{aligned}$$

◇ $\nabla\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot\nabla\times\mathbf{v} - \mathbf{v}\cdot\nabla\times\mathbf{w}$

$$\begin{aligned}
\nabla\cdot(\mathbf{v}\times\mathbf{w}) &= \nabla_i\epsilon_{ijk}v_j w_k = \frac{\partial}{\partial x_i}\epsilon_{ijk}v_j w_k = \epsilon_{ijk}w_k\left(\frac{\partial}{\partial x_i}v_j\right) + \epsilon_{ijk}v_j\left(\frac{\partial}{\partial x_i}w_k\right) \\
&= \epsilon_{ijk}w_k\left(\frac{\partial}{\partial x_i}v_j\right) - \epsilon_{jik}v_j\frac{\partial}{\partial x_i}w_k \\
&\overset{!}{=} \mathbf{w}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{w}) = w_i\epsilon_{ijk}\left(\frac{\partial}{\partial x_j}v_k\right) - v_i\epsilon_{ijk}\left(\frac{\partial}{\partial x_j}w_k\right) \\
&= \epsilon_{kij}w_k\left(\frac{\partial}{\partial x_i}v_j\right) - \epsilon_{jik}v_j\left(\frac{\partial}{\partial x_i}w_k\right) = \epsilon_{ijk}w_k\left(\frac{\partial}{\partial x_i}v_j\right) - \epsilon_{jik}v_j\left(\frac{\partial}{\partial x_i}w_k\right)
\end{aligned}$$

Where renaming of indices and even/odd permutations of the epsilon symbol were used (moving an index left or right by one position changes sign, moving an index by two position left or right does not)

◇ $\nabla\times(g\mathbf{A}) = g\nabla\times\mathbf{A} + \nabla g\times\mathbf{A}$

$$\begin{aligned}
(\nabla\times(g\mathbf{A}))_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(gA_k) = \epsilon_{ijk}\left(A_k\frac{\partial g}{\partial x_j} + g\frac{\partial A_k}{\partial x_j}\right) \\
&\overset{!}{=} (\nabla g\times\mathbf{A})_i + (g\nabla\times\mathbf{A})_i = \epsilon_{ijk}\frac{\partial g}{\partial x_j}A_k + g\epsilon_{ijk}\frac{\partial}{\partial x_j}A_k
\end{aligned}$$

◇ $\nabla\times(\nabla\times\mathbf{g})=\nabla(\nabla\cdot\mathbf{g})-(\nabla\cdot\nabla)\mathbf{g}$

$$\begin{aligned}(\nabla\times(\nabla\times\mathbf{g}))_i &= \epsilon_{ijk}\nabla_j(\nabla\times\mathbf{g})_k=\epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_l}g_m\right)=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_l}g_m\right)\\ &= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}g_j\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}g_i\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}g_j\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}g_i\right)\\ &= (\nabla(\nabla\cdot\mathbf{g}))_i-((\nabla\cdot\nabla)\mathbf{g})_i\end{aligned}$$

Here, Schwarz' lemma was used in the second last step.

◇ $\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$\begin{aligned}(\nabla\times(\mathbf{A}\times\mathbf{B}))_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}A_lB_m=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\left(B_m\frac{\partial A_l}{\partial x_j}+A_l\frac{\partial B_m}{\partial x_j}\right)\\ &= B_m\frac{\partial A_i}{\partial x_m}-B_i\frac{\partial A_j}{\partial x_j}+A_i\frac{\partial B_m}{\partial x_m}-A_l\frac{\partial B_i}{\partial x_l}\\ &\overset{!}{=} (\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B})_i\\ &= A_i\frac{\partial B_j}{\partial x_j}-B_i\frac{\partial A_j}{\partial x_j}+B_j\frac{\partial A_i}{\partial x_j}-A_j\frac{\partial B_i}{\partial x_j}\end{aligned}$$

◇ $\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$\begin{aligned}(\nabla(\mathbf{A}\cdot\mathbf{B}))_i &= \frac{\partial}{\partial x_i}\left(A_jB_j\right)=A_j\frac{\partial B_j}{\partial x_i}+B_j\frac{\partial A_j}{\partial x_i}\\ &\overset{!}{=} (\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{B})_i\\ &= \epsilon_{ijk}A_j\epsilon_{klm}\frac{\partial}{\partial x_l}B_m+\epsilon_{ijk}B_j\epsilon_{klm}\frac{\partial}{\partial x_l}A_m+B_j\frac{\partial}{\partial x_j}A_i+A_j\frac{\partial}{\partial x_j}B_i\\ &= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)A_j\frac{\partial B_m}{\partial x_l}+B_j\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)A_j\frac{\partial A_m}{\partial x_l}+B_j\frac{\partial A_i}{\partial x_j}+A_j\frac{\partial B_i}{\partial x_j}\\ &= A_m\frac{\partial B_m}{\partial x_i}-A_l\frac{\partial B_i}{\partial x_l}+B_m\frac{\partial A_m}{\partial x_i}-B_l\frac{\partial A_i}{\partial x_l}+B_j\frac{\partial A_i}{\partial x_j}+A_j\frac{\partial B_i}{\partial x_j}\\ &= A_m\frac{\partial B_m}{\partial x_i}+B_m\frac{\partial A_m}{\partial x_i}\end{aligned}$$

◇ $\nabla\cdot(f\mathbf{A})=\mathbf{A}\cdot\nabla f+f\nabla\cdot\mathbf{A}$

$$\nabla\cdot(f\mathbf{A}) = \frac{\partial}{\partial x_i}(fA_i)=A_i\frac{\partial f}{\partial x_i}+f\frac{\partial A_i}{\partial x_i}$$

◇ $\nabla(fg)=f\nabla g+g\nabla f$

$$(\nabla\cdot(fg))_i = \frac{\partial}{\partial x_i}(fg)=f\frac{\partial g}{\partial x_i}+g\frac{\partial f}{\partial x_i}$$

◇ $\Delta(hf)=f(\Delta h)+2(\nabla h)\cdot(\nabla f)+h(\Delta f)$

$$\begin{aligned}\Delta(hf) &= \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}(hf)=\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(f\frac{\partial h}{\partial x_i}+h\frac{\partial f}{\partial x_i}\right)\\ &= f\sum_{i=1}^{3}\frac{\partial^2 h}{\partial x_i^2}+\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right)+\sum_{i=1}^{3}\left(\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_i}\right)+h\sum_{i=1}^{3}\frac{\partial^2 f}{\partial x_i^2}\\ &= f(\Delta h)+(\Delta h)\cdot(\Delta f)+(\Delta h)\cdot(\Delta f)+h(\Delta f)\end{aligned}$$

6 Gauss Divergence Theorem variants

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To round this article, presented here are some standard textbook variants of Gauss' Divergence Theorem that may be found in many vector calculus textbooks.

◇ $\int_V \nabla\psi \mathrm{dV} = \int_{\partial V} \psi \mathrm{d}\mathbf{S}$

In the following two derivations **c** is taken to be a constant non-zero arbitrary vector.
We use the general form of **Gauss' Divergence theorem**:

$$\int_V \nabla\cdot\mathbf{A}\mathrm{dV} = \int_{\partial V} \mathbf{A}\cdot\mathrm{d}\mathbf{S} \tag{108}$$

Now use $\mathbf{A} = \mathbf{c}\psi$ in Gauss' Divergence Theorem and the previous vector identity $\nabla\cdot(\psi\mathbf{c}) = \mathbf{c}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{c}$:

$$\begin{aligned}\int_V \nabla\cdot(\mathbf{c}\psi)\,\mathrm{dV} &= \int_V (\mathbf{c}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{c})\,\mathrm{dV} = \int_V \mathbf{c}\cdot\nabla\psi\mathrm{dV} = \mathbf{c}\cdot\int_V \nabla\psi\mathrm{dV}\\ &= \int_{\partial V} \mathbf{c}\psi\cdot\mathrm{d}\mathbf{S} = \mathbf{c}\cdot\int_{\partial V}\psi\mathrm{d}\mathbf{S}\end{aligned}$$

This gives upon comparison of the last terms in both lines:

$$\mathbf{c}\cdot\left(\int_V \nabla\psi\mathrm{dV} - \int_{\partial V}\psi\mathrm{d}\mathbf{S}\right) = 0 \qquad \Leftrightarrow \qquad \int_V \nabla\psi\mathrm{dV} = \int_{\partial V}\psi\mathrm{d}\mathbf{S}$$

◇ $\int_V \nabla\times\mathbf{B}\mathrm{dV} = \int_{\partial V} \mathrm{d}\mathbf{S}\times\mathbf{B}$

On the other hand, using $\mathbf{A} = \mathbf{c}\times\mathbf{B}$ in the Divergence Theorem and the vector identity $\nabla\cdot(\mathbf{c}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{c} - \mathbf{c}\cdot\nabla\times\mathbf{B}$ we obtain the following result:

$$\begin{aligned}\int_V \nabla\cdot(\mathbf{c}\times\mathbf{B})\,\mathrm{dV} &= \int_V (\mathbf{B}\cdot(\nabla\times\mathbf{c}) - \mathbf{c}\cdot(\nabla\times\mathbf{B}))\mathrm{dV} = -\mathbf{c}\cdot\int_V \nabla\times\mathbf{B}\mathrm{dV}\\ &= \int_{\partial V}\mathbf{c}\times\mathbf{B}\cdot\mathrm{d}\mathbf{S} = \mathbf{c}\cdot\int_{\partial V}\mathbf{B}\times\mathrm{d}\mathbf{S} = -\mathbf{c}\cdot\int_{\partial V}\mathbf{n}\times\mathbf{B}\mathrm{dS}\end{aligned}$$

In the last step I used cyclic permutations of the mixed product. Comparison of the last elements of both lines, and noting the arbitrariness of the constant vector **c** we get the form we wanted to proof.

◇ $\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,\mathrm{dV} = \int_{\partial V}\phi\nabla\psi\cdot\mathrm{d}\mathbf{S}$

To proove this identity (**Green's first identity**) simply put $\mathbf{A} = \phi\nabla\psi$ into Gauss' Divergence Theorem:

$$\int_V \nabla\cdot(\phi\nabla\psi)\,\mathrm{dV} = \int_V \left(\phi\nabla^2\psi + \nabla\phi\nabla\psi\right)\mathrm{dV} = \int_{\partial V}\phi\nabla\psi\cdot\mathrm{dS}$$

◇ $\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,\mathrm{dV} = \int_{\partial V}(\phi\nabla\psi - \psi\nabla\phi)\cdot\mathrm{d}\mathbf{S}$

To proove Green's theorem we interchange the role of ϕ and ψ and substract them from each other.

$$\int_V \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi - \psi\nabla^2\phi - \nabla\psi\cdot\nabla\phi\right)\mathrm{dV} = \int_{\partial V}(\phi\nabla\psi - \psi\nabla\phi)\cdot\mathrm{d}\mathbf{S}$$

Summarizing we have found and prooved the following product rules:

$$\begin{aligned}
\nabla(fg) &= f\nabla g + g\nabla f && (109)\\
\nabla\cdot(f\mathbf{A}) &= \mathbf{A}\cdot\nabla f + f\nabla\cdot\mathbf{A} && (110)\\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} && (111)\\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} && (112)\\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - (\nabla\cdot\nabla)\mathbf{A} && (113)\\
\nabla\times(g\mathbf{A}) &= g\nabla\times\mathbf{A} + \nabla g\times\mathbf{A} && (114)\\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) && (115)\\
(\mathbf{B}\cdot\nabla)\mathbf{A} &= \frac{1}{2}\left[\nabla\times(\mathbf{A}\times\mathbf{B}) + \nabla(\mathbf{A}\cdot\mathbf{B}) - \mathbf{A}(\nabla\cdot\mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{A}) - \mathbf{A}\times(\nabla\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{A})\right] && (116)\\
\Delta(hf) &= f(\Delta h) + 2(\nabla h)\cdot(\nabla f) + h(\Delta f) && (117)\\
\nabla^2\mathbf{A} &= \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) && (118)
\end{aligned}$$

$$\nabla$$

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