Stirling's Approximation

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1 Stirling's Approximation

$$\begin{aligned}
\ln(n!) &= \ln(n)+\ln(n-1)+\ldots+\ln(1)=\sum_{k=1}^{n}\ln(k)=\sum_{k=1}^{n}\ln(k)\times((k+1)-k)=\sum_{k=1}^{n}\ln(k)\Delta k \quad (1)\\
&\approx \int_{k=1}^{k=n}\ln(k)\delta k=[k\ln(k)-k]_1^n=n\ln(n)-n-(1\ln(1)-1)=n\ln(n)-n+1\\
&\approx n\ln(n)-n \qquad \text{for large } n \quad (2)\\
n! &\approx e^{n\ln(n)-n+1}=n^n e^{-n+1} \quad (3)
\end{aligned}$$

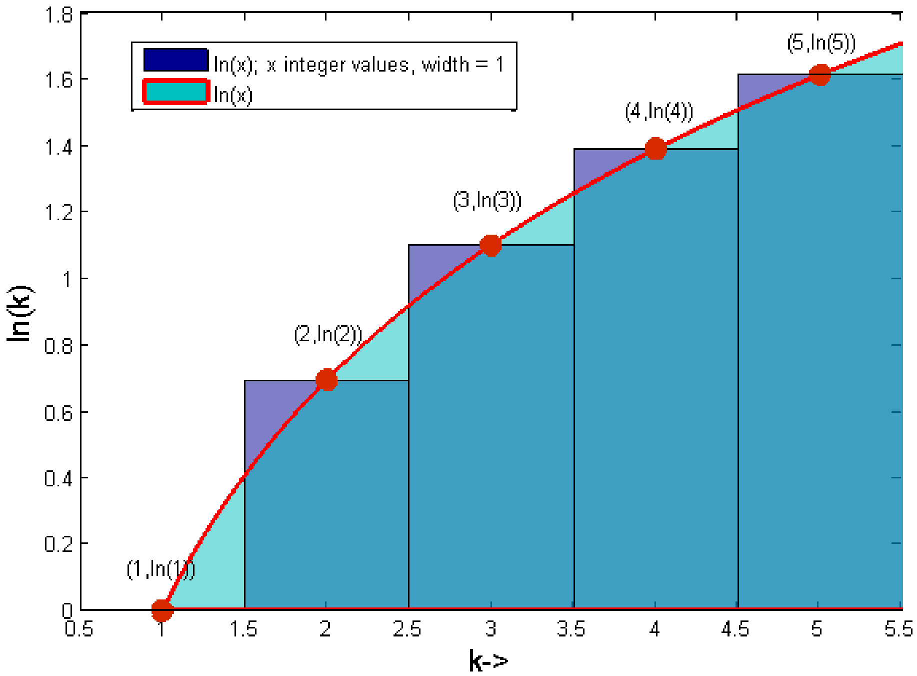

Figure 1: The Integration as an appoximation for the actual summation

We will use a more **precise form of Sterlings Approximation** though. To do so we will use the expansion of the factorial to the real line, namely the Gamma function:

$$\begin{aligned}
n! &= \Gamma(n+1)\\
\Gamma(n) &= \int_0^\infty t^{n-1}e^{-t}\mathrm{d}t\\
n! &= \int_0^\infty t^n e^{-t}\mathrm{d}t \quad (4)
\end{aligned}$$

further we note the following: The derivative of the logarithm of the integrand is

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(t^n e^{-t}\right) = \frac{-e^{-t}t^n + nt^{n-1}e^{-t}}{e^{-t}t^n} = -1 + \frac{n}{t} \tag{5}$$

Hence we see that the integrand is sharply peaked at $t \approx n$. Here we expect the highest contribution to the integral. Substitution of variables $t = n + \epsilon$ with $\epsilon \ll n$ then gives:

$$\begin{aligned}
\ln\left(t^n e^{-t}\right) &= n\ln(t) - t = n\ln(n+\epsilon) - (n+\epsilon) && (6)\\
\ln(n+\epsilon) &= \ln\left(n\left(1+\frac{\epsilon}{n}\right)\right) = \ln n + \ln\left(1+\frac{\epsilon}{n}\right) = \ln n + \frac{\epsilon}{n} - \frac{1}{2}\frac{\epsilon^2}{n^2} + \frac{1}{3}\frac{\epsilon^3}{n^3} - \ldots \\
n\ln(n+\epsilon) &= n\ln n + \epsilon - \frac{1}{2}\frac{\epsilon^2}{n} + \ldots && (7)
\end{aligned}$$

So we get the following expression when we substitute eqn (7) in (6):

$$\ln\left(t^n e^{-t}\right) \approx n\ln n + \epsilon - \frac{1}{2}\frac{\epsilon^2}{n} + \ldots - n - \epsilon \approx n\ln n - n - \frac{\epsilon^2}{2n} \tag{8}$$

Now we take the exponential on both sides of equation (8) and plug it into (4) to obtain:

$$\begin{aligned}
t^n e^{-t} &\approx e^{n\ln n}e^{-n}e^{-\frac{\epsilon^2}{2n}} = n^n e^{-n} e^{-\frac{\epsilon^2}{2n}} \\
n! &\approx \int_{-n}^{\infty} n^n e^{-n} e^{-\frac{\epsilon^2}{2n}}\,\mathrm{d}\epsilon \qquad \text{where } \epsilon = t - n \text{ was used for the boundaries} \\
&\approx n^n e^{-n}\int_{-n}^{\infty} e^{-\frac{\epsilon^2}{2n}}\,\mathrm{d}\epsilon \approx n^n e^{-n}\int_{-\infty}^{\infty} e^{-\frac{\epsilon^2}{2n}}\,\mathrm{d}\epsilon \\
&= n^n e^{-n}\sqrt{\frac{\pi}{1/2n}} = n^n e^{-n}\sqrt{2\pi n} = n^{n+1/2}e^{-n}\sqrt{2\pi} && (9)\\
\ln(n!) &\approx \ln\left(n^{n+1/2}e^{-n}\sqrt{2\pi}\right) = (n+1/2)\ln n - n + \frac{1}{2}\ln(2\pi) && (10)
\end{aligned}$$

If you compare equation (10) with (2) which was found the easy way you see that our more sophisticated one reduces to the (10) in the limit where n is large.

$$n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi} = n^n e^{-n}\sqrt{2\pi n} \tag{11}$$