Bernoulli Numbers

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Contents

Abstract

This Document will introduce a generating function for the Bernoulli numbers and give an explicit recursion formula based on it. The first few will be calculated as an example. In the following sections steps will be taken in order to get a closed formula for the even-valued Riemann Zeta function. Then some applications for the Zeta function are formulated including an informal prove for the Euler pruduct formula. Finally another calculation involving the Bernoulli numbers and its generating function will lead to a closed formula for the finite sum of exponentials.

1 The Bernoulli Numbers

1.1 The Generating Function

By generating function usually a function is meant, whose Taylor expansion yields the seeked for expression. In our case it will be a simple fraction:

$$f(x) = \frac{x}{e^x - 1} \quad \forall x \neq 0, \quad f^n(0) = 0, \quad \forall n = 0, 1, \ldots, \quad n \in \mathbb{N}, \tag{1}$$

The Bernoulli numbers are defined to be the coefficients of the taylor expansion for this function. However, to obtain them we have to go a different way than usually.

$$f(x) := \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{2}$$

by definition we have the taylor series for the exponential function:

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{3}$$

$$f(x) = \frac{1}{\left(\sum_{k=0}^{\infty} \frac{x^k}{k!} - 1\right) \frac{1}{x}} = \left(\sum_{k=1}^{\infty} \left(\frac{x^k}{k!}\right) \frac{1}{x}\right)^{-1} = \left(\sum_{k=1}^{\infty} \left(\frac{x^{k-1}}{k!}\right)\right)^{-1} = \left(\sum_{k=0}^{\infty} \left(\frac{x^k}{(k+1)!}\right)\right)^{-1} \tag{4}$$

$$= \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{5}$$

$$\Rightarrow \quad \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n\right) \left(\sum_{k=0}^{\infty} \left(\frac{x^k}{(k+1)!}\right)\right) = 1 \tag{6}$$

Now we make use of the Cauchy Product of series (see Appendix A):

$$1 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k x^k}{k!} \frac{x^{n-k}}{(n-k+1)!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k \frac{x^n}{k!(n+1-k)!}\right) \tag{7}$$

We use the definition of n choose k to rewrite this (see Appendix B):

$$1 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k \binom{n+1}{k}\right) \frac{x^n}{(n+1)!} \tag{8}$$

This equation is somewhat equivalent to a binomial expression if one defines the exponent of B to be understood as a subscript (Then compare eqn. (10) with (8)):

$$(B+1)^{n+1} - B^{n+1} = \sum_{k=0}^{n+1} B^k 1^{n+1-k} \binom{n+1}{k} - B^{n+1} \tag{9}$$

$$= \sum_{k=0}^{n} B^k \binom{n+1}{k} + B^{n+1} \binom{n+1}{n+1} - B^{n+1} = \sum_{k=0}^{n} B^k \binom{n+1}{k} \tag{10}$$

1.2 The Recurrence Formula

Now the procedure of comparison of coefficients yields the first Bernoulli number B_0:

$$1 = \sum_{k=0}^{0} B_k \binom{0+1}{k} \frac{x^0}{(0+1)!} = B_0 \frac{1!}{0!1!} \frac{1!}{1!} \quad \Rightarrow B_0 = 1 \tag{11}$$

For the following numbers we use a **recurrence formula** by noting that comparison of coefficients tells us: For all n greater 0, the sum in the brackets of equation (8) has to vanish!

$$\sum_{k=0}^{n} B_k \binom{n+1}{k} = 0 \quad \forall\, n \geq 1, \quad n \in \mathbb{N}, \tag{12}$$

A nice interpretation of this formula is sugested by the similarity to the Binomial expansion introduced in equation (9) and (10). We only have to write down the appropriate row of the Pascal Triangle for $(B+1)^{n+1}$ without the last term

$$\begin{array}{lcl}
n=1 & 0 = B_0 + B_1 & \Rightarrow B_1 = -\frac{1}{2} \\
n=2 & 0 = B_0 + 3B_1 + 3B_2 & \Rightarrow B_2 = \frac{1}{6} \\
n=3 & 0 = B_0 + 4B_1 + 6B_2 + 4B_3 & \Rightarrow B_3 = 0 \\
n=4 & 0 = B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 & \Rightarrow B_4 = -\frac{1}{30} \\
n=5 & 0 = B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5 & \Rightarrow B_5 = 0 \\
n=6 & 0 = B_0 + 7B_1 + 21B_2 + 35B_3 + 35B_4 + 21B_5 + 7B_6 & \Rightarrow B_6 = \frac{1}{42}
\end{array}$$

n	0	1	2	4	6	8	10	12	14	16	18	20
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

Table 1: some Bernoulli Numbers, note: $B_{2n+1} = 0 \quad \forall n \geq 1$

1.3 Stirling's Approximation and Approximation of B_{2k}

$$\ln(n!) = \ln(n) + \ln(n-1) + \ldots + \ln(1) = \sum_{k=1}^{n} \ln(k) = \sum_{k=1}^{n} \ln(k) \times ((k+1)-k) = \sum_{k=1}^{n} \ln(k)\Delta k \tag{13}$$

$$\ln(n!) \approx \int_{k=1}^{k=n} \ln(k)\,\delta k = [k\ln(k) - k]_1^n = n\ln(n) - n - (1\ln(1) - 1) = n\ln(n) - n + 1$$

$$\ln(n!) \approx n\ln(n) - n \qquad \text{for large } n \tag{14}$$

$$n! \approx e^{n\ln(n)-n+1} = n^n e^{-n+1} \tag{15}$$

We will use a more **precise form of Sterlings Approximation** though. To do so we will use the expansion of the factorial to the real line, namely the Gamma function:

$$\begin{aligned}
n! &= \Gamma(n+1) \\
\Gamma(n) &= \int_0^\infty t^{n-1} e^{-t} \mathrm{d}t \\
n! &= \int_0^\infty t^n e^{-t} \mathrm{d}t
\end{aligned} \tag{16}$$

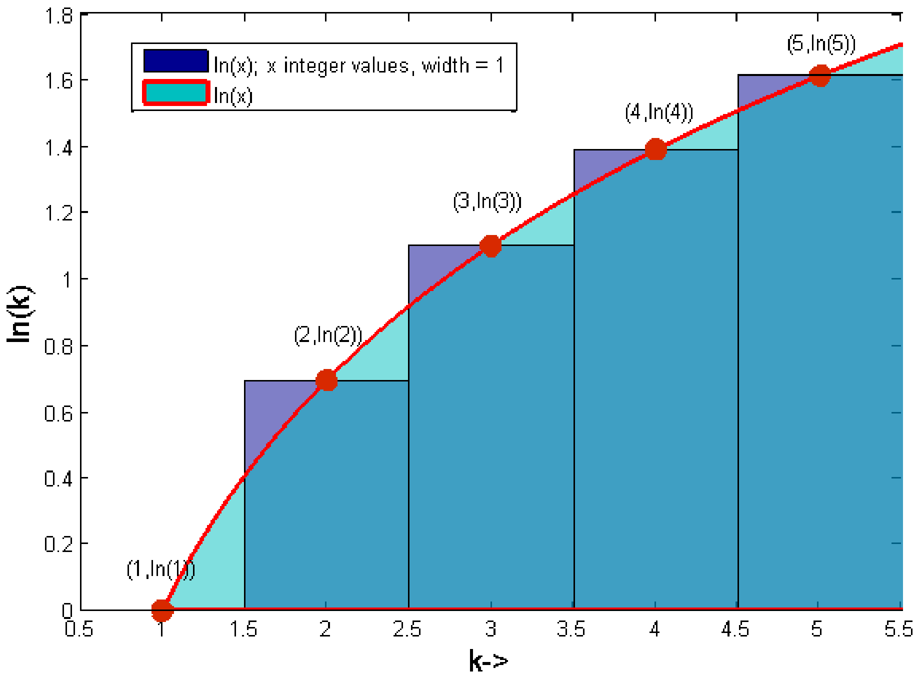


Figure 1: The Integration as an appoximation for the actual summation

further we note the following: The derivative of the logarithm of the integrand is

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(t^n e^{-t}\right) = \frac{-e^{-t}t^n + nt^{n-1}e^{-t}}{e^{-t}t^n} = -1 + \frac{n}{t} \tag{17}$$

Hence we see that the integrand is sharply peaked at $t \approx n$. Here we expect the highest contribution to the integral. Substitution of variables $t = n + \epsilon$ with $\epsilon \ll n$ then gives:

$$\ln\left(t^n e^{-t}\right) = n\ln(t) - t = n\ln(n+\epsilon) - (n+\epsilon) \tag{18}$$

$$\begin{aligned}
\ln(n+\epsilon) &= \ln\left(n\left(1+\frac{\epsilon}{n}\right)\right) = \ln n + \ln\left(1+\frac{\epsilon}{n}\right) = \ln n + \frac{\epsilon}{n} - \frac{1}{2}\frac{\epsilon^2}{n^2} + \frac{1}{3}\frac{\epsilon^3}{n^3} - \dots \\
n\ln(n+\epsilon) &= n\ln n + \epsilon - \frac{1}{2}\frac{\epsilon^2}{n} + \dots
\end{aligned} \tag{19}$$

So we get the following expression when we substitute eqn (19) in (18):

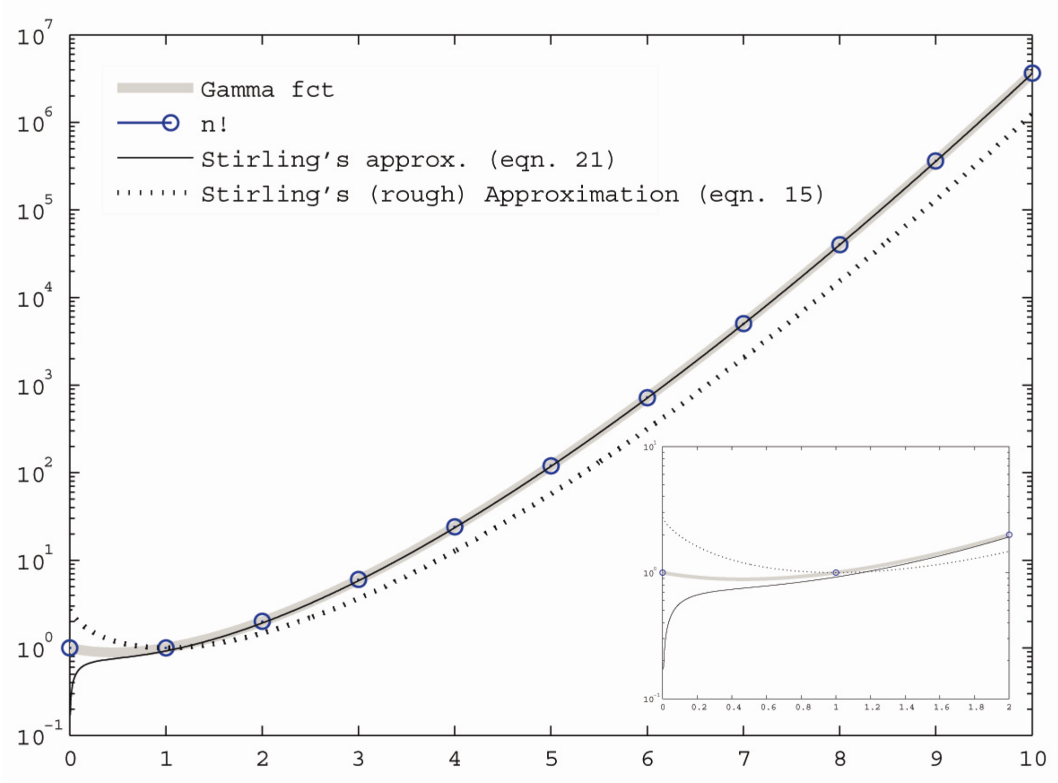
$$\ln\left(t^n e^{-t}\right) \approx n\ln n + \epsilon - \frac{1}{2}\frac{\epsilon^2}{n} + \dots - n - \epsilon \approx n\ln n - n - \frac{\epsilon^2}{2n} \tag{20}$$

Now we take the exponential on both sides of equation (20) and plug it into (16) to obtain:

$$\begin{aligned}
t^n e^{-t} &\approx e^{n\ln n} e^{-n} e^{-\frac{\epsilon^2}{2n}} = n^n e^{-n} e^{-\frac{\epsilon^2}{2n}} \\
n! &\approx \int_{-n}^{\infty} n^n e^{-n} e^{-\frac{\epsilon^2}{2n}} \mathrm{d}\epsilon \qquad \text{where } \epsilon = t-n \text{ was used for the boundaries} \\
&\approx n^n e^{-n} \int_{-n}^{\infty} e^{-\frac{\epsilon^2}{2n}} \mathrm{d}\epsilon \approx n^n e^{-n} \int_{-\infty}^{\infty} e^{-\frac{\epsilon^2}{2n}} \mathrm{d}\epsilon \\
&= n^n e^{-n} \sqrt{\frac{\pi}{1/2n}} = n^n e^{-n}\sqrt{2\pi n} = n^{n+1/2} e^{-n}\sqrt{2\pi}
\end{aligned} \tag{21}$$

$$\ln(n!) \approx \ln\left(n^{n+1/2} e^{-n}\sqrt{2\pi}\right) = (n+1/2)\ln n - n + \frac{1}{2}\ln(2\pi) \tag{22}$$

If you compare equation (22) with (14) which was found the easy way you see that our more sophisticated one reduces to the (14) in the limit where n is large.



We will use equation (21) to obtain an **approximation for the Bernoulli numbers**!

$$(2k)! \approx (2k)^{2k+1/2}\, e^{-2k}\sqrt{2\pi} = (2k)^{2k}\, e^{-2k}\sqrt{4\pi k} \tag{23}$$

Using equation (76) we get:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} &= (-1)^{k+1}(2\pi)^{2k}\frac{B_{2k}}{2(2k)!} = \zeta(2k) \\ &\Rightarrow \\ B_{2k} &= (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k) \approx (-1)^{k+1}\frac{2(2k)^{2k}\, e^{-2k}\sqrt{4\pi k}}{(2\pi)^{2k}}\zeta(2k) \end{aligned} \tag{24}$$

(25)

If we seek for appoximate coefficients of the B_{2k}'s we consider the limit of $\zeta(2k)$ as k grows:

$$\lim_{k\to\infty} \zeta(2k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 1 \tag{26}$$

Since in this limit only the first term of the series, which is equal to one, survives.
Hence we finally obtain for our estimate for B_{2k}:

$$B_{2k} \approx (-1)^{k+1}\frac{2(2k)^{2k}\, e^{-2k}\sqrt{4\pi k}}{(2\pi)^{2k}} = 4k^{2k+1/2}e^{-2k}\pi^{1/2-2k} = 4\left(\frac{k}{\pi e}\right)^{2k}\sqrt{k\pi} \tag{27}$$

Also knowing the following functional relation, the **reflection formula**, we can compute $\zeta(1-2n)$:[1]

$$\begin{aligned} \zeta(x) &= 2^x \pi^{x-1}\sin\left(\frac{1}{2}x\pi\right)\Gamma(1-x)\zeta(1-x) \\ x = 1-2n \quad &\Rightarrow \quad \zeta(1-2n) = -\frac{B_{2n}}{2n} \end{aligned}$$

[1] proof can be found here: http://scipp.ucsc.edu/ haber/ph116A/pibern.pdf

1.4 Some Series using the Bernoulli Numbers

In this section we will use equation (50) and (63) which will be derived in the next section in detail. However we will have to modify them slightly by substituting x for $\frac{x'}{2}$ in equation (50) ($\Rightarrow x' = 2x$):

$$\frac{x'}{2}\coth\frac{x'}{2} = 1 + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}x'^{2n} = x\coth x = 1 + \sum_{n=1}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n} \tag{28}$$

To transform this series representation to the regular cotangens we use quation (63) with $y' = iy$

$$\cot(-y) = -\cot(y) = -i\coth(iy) \quad \Rightarrow y\cot(y) = (iy)\coth(iy) \tag{29}$$

$$x\cot x = \sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n}\left(i^2\right)^n = \sum_{n=0}^{\infty}\frac{4^n(-1)^n B_{2n}}{(2n)!}x^{2n} \tag{30}$$

Dividing by x we get the following two series expressions for $\cot x$ and $\coth x$:

$$\coth x = \sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n-1} \tag{31}$$

$$\cot x = \sum_{n=0}^{\infty}\frac{4^n(-1)^n B_{2n}}{(2n)!}x^{2n-1} \tag{32}$$

We will use them and additionally some trigonometric identities to derive the Taylor expansion of the tangens function and the tangens hyperbolicus.

$$-\tan x = 2\cot(2x) - \cot(x) = 2\frac{\cos(2x)}{\sin(2x)} - \frac{\cos(x)}{\sin(x)} = \frac{2\cos(x)\cos(x) - 2\sin(x)\sin(x)}{2\sin(x)\cos(x)} - \frac{\cos(x)}{\sin(x)} \tag{33}$$

$$\tan x = (-1)\left[2\sum_{n=0}^{\infty}\frac{4^n(-1)^n B_{2n}}{(2n)!}x^{2n-1}2^{2n-1} - \sum_{n=0}^{\infty}\frac{4^n(-1)^n B_{2n}}{(2n)!}x^{2n-1}\right] = \sum_{n=0}^{\infty}\frac{4^n(4^n-1)(-1)^{n+1}B_{2n}}{(2n)!}x^{2n-1} \tag{34}$$

Using a similar relation for the tangens hyperbolicus we obtain the corresponding series expression:

$$\tanh x = 2\coth(2x) - \coth(x) = 2\frac{\cosh(2x)}{\sinh(2x)} - \frac{\cosh(x)}{\sinh(x)} = \frac{2\cosh(x)\cosh(x) + 2\sinh(x)\sinh(x)}{2\sinh(x)\cosh(x)} - \frac{\cosh(x)}{\sinh(x)} \tag{35}$$

$$\tanh x = 2\sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n-1}2^{2n-1} - \sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n-1} = \sum_{n=0}^{\infty}\frac{4^n(4^n-1)B_{2n}}{(2n)!}x^{2n-1} \tag{36}$$

Summarizing we have:

$\cot x$	$\coth x$	$\tan x$	$\tanh x$
$\sum_{n=0}^{\infty}\frac{4^n(-1)^n B_{2n}}{(2n)!}x^{2n-1}$	$\sum_{n=0}^{\infty}\frac{4^n B_{2n}}{(2n)!}x^{2n-1}$	$\sum_{n=1}^{\infty}\frac{4^n(4^n-1)(-1)^{n+1}B_{2n}}{(2n)!}x^{2n-1}$	$\sum_{n=1}^{\infty}\frac{4^n(4^n-1)B_{2n}}{(2n)!}x^{2n-1}$
$\frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \dots$	$\frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \dots$	$x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$	$x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$

Table 2: some series representations of trigonometric functions. ($\csc x = \cot x/2 - \cot x$ also involves B_{2n})

Now we use the general formula of the series of the $\sin x$ function to obtain a series representation of the $\arcsin x$

$$\sin x = \sum_{n=0}^{\infty}\frac{(\sin x)^{(n)}|_{x=0}x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty}\frac{(\cos x)^{(n)}|_{x=0}x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \tag{37}$$

We first want to find the integral of $\arcsin x$ using the following property:

Theorem 1 *Let $f:(a,b)\to\mathbb{R}$ be strictly monotonic and continuous. Suppose that f is differentiable at $x\in(a,b)$. Then the inverse function $g=f^{-1}:\ f((a,b))\to\mathbb{R}$ is differentiable at $y=f(x)$ with*
$g'(y)=\frac{1}{f'(x)}=\frac{1}{f'(g(y))}$

$$\begin{aligned} f(x) &= \sin(x)=y \qquad g(y)=\arcsin(y) \\ g'(y) &= \frac{1}{f'(x)}=\frac{1}{f'(g(y))}=\frac{1}{\cos(x)}=\frac{1}{\sqrt{1-(\sin(x))^2}}=\frac{1}{\sqrt{1-y^2}} \end{aligned} \tag{38}$$

We start to construct the Taylor series around $a=0$ of the derivative of $\arcsin(x)$:

$$\begin{array}{cccc} f(x)=\dfrac{1}{\sqrt{1-x}} & f(0)=1 \quad ; & f^{(1)}(x)=\dfrac{1}{2(1-x)^{1/2}} & f^{(1)}(0)=\dfrac{1}{2} \\ f^{(2)}(x)=\dfrac{1\times3}{2(1-x)^{3/2}} & f^{(2)}(0)=\dfrac{1\times3}{2\times2} \quad ; & f^{(3)}(x)=\dfrac{1\times3\times5}{2(1-x)^{5/2}} & f^{(3)}(0)=\dfrac{1\times3\times5}{2\times2\times2} \end{array} \tag{39}$$

At this point we note the following system in the derivatives:

$$\frac{5\times3\times1}{2\times2\times2}=\frac{5\times4\times3\times2\times1}{2^3\times3!\times2\times2\times2}=\frac{6!}{3!(2^3)^2} \tag{40}$$

Hence we get the general series expression and, afterwards, by substituting x by x^2:

$$\frac{1}{\sqrt{1-x}}=\sum_{k=0}^{\infty}\frac{(2k)!x^k}{(2^k)^2(k!)^2} \qquad \Rightarrow \qquad \frac{1}{\sqrt{1-x^2}}=\sum_{k=0}^{\infty}\frac{(2k)!x^{2k}}{2^{2k}(k!)^2} \tag{41}$$

...and finally we do the following trick:

$$\arcsin(y)=\int\sum_{k=0}^{\infty}\frac{(2k)!x^{2k}}{2^{2k}(k!)^2}\mathrm{d}x=\sum_{k=0}^{\infty}\frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \tag{42}$$

A similar technique gives us the formula for the $\arctan x$:

$$\begin{aligned} f(x) &= \tan(x) \qquad g(y)=\arctan(y) \\ g'(y) &= \frac{1}{f'(x)}=(\cos x)^2=\frac{1}{(\tan x)^2+1}=\frac{1}{1+y^2} \end{aligned} \tag{43}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty}x^k \quad \text{(Geometric Series)} \quad \Rightarrow \quad \frac{1}{1+x^2}=\sum_{k=0}^{\infty}\left(-x^2\right)^k=\sum_{k=0}^{\infty}x^{2k}(-1)^k \tag{44}$$

$$\arctan x = \int\sum_{k=0}^{\infty}x^{2k}(-1)^k=\sum_{k=0}^{\infty}\frac{x^{2k+1}(-1)^k}{(2k+1)} \tag{45}$$

We now complete this discussion by the following two series:

$$\arccos x = \frac{\pi}{2}-\arcsin x=\frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)} \tag{46}$$

$$\operatorname{arccot} x = \frac{\pi}{2}-\arctan x=\frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{x^{2k+1}(-1)^k}{(2k+1)} \tag{47}$$

$\arcsin x$	$\arccos x$	$\arctan x$	$\operatorname{arccot} x$
$\sum_{k=0}^{\infty}\frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)}$	$\frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{(2k)!x^{2k+1}}{2^{2k}(k!)^2(2k+1)}$	$\sum_{k=0}^{\infty}\frac{x^{2k+1}(-1)^k}{(2k+1)}$	$\frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{x^{2k+1}(-1)^k}{(2k+1)}$
$x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7+\ldots$	$\frac{\pi}{2}-x-\frac{1}{6}x^3-\frac{3}{40}x^5-\ldots$	$x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots$	$\frac{\pi}{2}-x+\frac{1}{3}x^3-\frac{1}{5}x^5+\ldots$

2 The Zeta Function

The aim will be to express the Riemann Zeta function with the help of the Bernoulli numbers. To do so we will find two notions of one and the same expression and eventually compare both to obtain the result.

2.1 First Notion of $\frac{x}{2}\coth\frac{x}{2}$

$$\frac{x}{2}\coth\frac{x}{2} = \frac{x}{2}\frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}} = \frac{x}{2}\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} \times \overbrace{\frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}}}^{=1} \tag{48}$$

$$= \frac{x}{2}\frac{e^x+1}{e^x-1} = \frac{x}{e^x-1}+\frac{x}{2} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n+\frac{x}{2} \tag{49}$$

If we now note that $B_0 = 1$ and $B_1 = -1/2$ we get

$$\frac{x}{2}\coth\frac{x}{2} = \frac{1x^0}{0!} - \frac{x^1}{2\times 1!} + \sum_{n=2}^{\infty}\frac{B_n}{n!}x^n + \frac{x}{2} = 1+\sum_{n=2}^{\infty}\frac{B_n}{n!}x^n \tag{50}$$

Now it also gets clear why table 1 only lists even subscripts of Bernoulli numbers. The cotangens hyperbolicus, being a quotient of an even and an odd function, is odd again. x is an odd function as well, and hence $\frac{x}{2}\coth\frac{x}{2}$ is an even function. Therefore, by this representation of our series we note that $B_{2n+1} = 0$ for all n greater or equal 1. ($B_{2n+1} = 0 \quad \forall n = 1, 2, \ldots$)
Therefore we can write down more succintly:

$$\frac{x}{2}\coth\frac{x}{2} = 1+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n} \tag{51}$$

2.2 Second Notion of $\frac{x}{2}\coth\frac{x}{2}$

2.2.1 Fourier Series of $\cos(yx)$

In this section we will calculate the Fourier Series of the following function g(x). Later we will take a special value for x in the found expression to continue:

$$g(x) = cos\,(yx)\,, \quad y \in \mathbb{R}\backslash\mathbb{Z}, \quad \mathbb{D} = [-\pi, \pi] \tag{52}$$

The general formula of the Fourier series is:

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx) \tag{53}$$

Where it saves a lot of time to note that the coefficients b_k are all zero, since our founction is even! I will also use the following relations during the calculation of a_k:

$$\cos x\cos y = \frac{1}{2}\left[\cos(x-y)\cos(x+y)\right], \qquad \sin(-x) = -\sin(x)$$

$$\sin(a+b) = \sin(a)\cos(b)+\cos(a)\sin(b), \quad \sin(k\pi) = 0, \quad \cos(k\pi) = (-1)^k$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi}\cos yx\cos kx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{+\pi}\frac{1}{2}\left(\cos(yx-kx)\cos(yx+kx)\right)\mathrm{d}x \tag{54}$$

$$= \frac{1}{2\pi}\left[\frac{\sin((y-k)x)}{y-k}+\frac{\cos((y+k)x)}{y+k}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\sin((y-k)\pi)}{y-k}+\frac{\cos((y+k)\pi)}{y+k}\right) \tag{55}$$

$$= \frac{1}{\pi}\left(\frac{(y+k)\left(\sin y\pi\cos k\pi - \sin k\pi\cos y\pi\right)+(y-k)\left(\sin y\pi cos k\pi + \sin k\pi\cos y\pi\right)}{(y+k)(y-k)}\right) \tag{56}$$

$$= \frac{1}{\pi}\left(\frac{y\sin(y\pi)(-1)^k + k\sin(y\pi)(-1)^k + y\sin(y\pi)(-1)^k - k\sin(y\pi)(-1)^k}{y^2-k^2}\right) \tag{57}$$

$$= \frac{1}{\pi}\left(\frac{2y\sin(y\pi)(-1)^k}{y^2-k^2}\right) \tag{58}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi}\cos(yx)\mathrm{d}x = \frac{1}{\pi}\left[\frac{1}{y}\sin(yx)\right]_{-\pi}^{+\pi} = \frac{1}{\pi}\left(\frac{2}{y}\sin(y\pi)\right) \tag{59}$$

Plugging these coefficients back into equation (53) gives our Fourier Series of g(x):

$$\cos(yx) = \frac{\sin(y\pi)}{y\pi} + \sum_{k=1}^{\infty}\frac{1}{\pi}\left(\frac{2y\sin(y\pi)(-1)^k}{y^2-k^2}\right)\cos(kx) \tag{60}$$

Now we consider the special case $x = \pi$:

$$\cos(y\pi) = \frac{\sin(y\pi)}{\pi}\left[\frac{1}{y} + \sum_{k=1}^{\infty}\left(\frac{2y(-1)^k}{y^2-k^2}\right)(-1)^k\right] \tag{61}$$

$$\Rightarrow \pi\times\frac{\cos(y\pi)}{\sin(y\pi)} = \frac{1}{y} + \sum_{k=1}^{\infty}\frac{2y}{y^2-k^2} = \pi\cot(y\pi) \tag{62}$$

This is the so-called **partial fraction decomposition** of the cotangens function.

2.2.2 Transformation to $\frac{x}{2}\coth\frac{x}{2}$

We will now transform the regular cotangens and its partial fraction decomposition into the second form of the cotangens hyperbolicus we need for the comparison ahead.

$$\cot(iy') = \frac{\frac{1}{2}\left(e^{i\times iy'} + e^{-i\times iy'}\right)}{\frac{1}{2i}\left(e^{i\times iy'} - e^{-i\times iy'}\right)} = i\frac{\left(e^{-y'} + e^{y'}\right)}{\left(e^{-y'} - e^{y'}\right)} = -i\coth(y') \tag{63}$$

We multiply this equation by $i\pi$ and let $y' = y\pi$ to fit our purposes:

$$i\pi\cot(iy') = \pi\coth(y') \Rightarrow i\pi\cot(iy\pi) = \pi\coth(y\pi) \tag{64}$$

$$\pi\coth(y\pi) = i\left[\frac{1}{iy} + \sum_{k=1}^{\infty}\frac{2iy}{(iy)^2-k^2}\right] = \frac{1}{y} + \sum_{k=1}^{\infty}\frac{-2y}{-y^2-k^2} = \frac{1}{y} + \sum_{k=1}^{\infty}\frac{2y}{y^2+k^2} \tag{65}$$

Now we will do the following substitution to get the right arguement in the cotangens hyperbolicus:

$$y\pi = \frac{x}{2} \quad\Rightarrow\quad y = \frac{x}{2\pi} \quad\Rightarrow\quad \pi = \frac{x}{2y} \tag{66}$$

$$\frac{x}{2y}\coth\left(\frac{x}{2}\right) = \frac{1}{y} + \sum_{n=1}^{\infty}\frac{2y}{\frac{x^2}{4\pi^2}+n^2} \tag{67}$$

$$\Rightarrow \frac{x}{2}\coth\left(\frac{x}{2}\right) = 1 + \sum_{n=1}^{\infty}\frac{2\frac{x^2}{4\pi^2}}{\frac{x^2}{4\pi^2}+n^2} = 1 + \sum_{n=1}^{\infty}\frac{2x^2}{x^2+4\pi^2n^2} \tag{68}$$

We need to recognize that this infinite sum represents a geometric series (see Appendix C)

$$-\sum_{k=1}^{\infty}\left(-\frac{x^2}{4\pi^2n^2}\right)^k \quad = \quad -\frac{-\frac{x^2}{4\pi^2n^2}}{1+\frac{x^2}{4\pi^2n^2}} = \frac{x^2}{x^2+4\pi^2n^2} \tag{69}$$

$$= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{x}{2\pi n}\right)^{2k} \tag{70}$$

$$\Rightarrow \frac{x}{2}\coth\left(\frac{x}{2}\right) = 1 + 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{x}{2\pi n}\right)^{2k} = 1 + 2\sum_{k=1}^{\infty}\left[(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^{2k}}\right]x^{2k} \tag{71}$$

2.3 Comparison and Result

Now we compare our found equations (71) and (51)

$$\frac{x}{2}\coth\left(\frac{x}{2}\right) = 1+2\sum_{k=1}^{\infty}\left[(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^{2k}}\right]x^{2k} \quad (72)$$

$$= 1+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n} \quad (73)$$

$$\Rightarrow \sum_{k=1}^{\infty}\left[(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^{2k}}\right]x^{2k} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}x^{2k} \quad (74)$$

Comparison of coefficients yields:

$$(-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{(2\pi n)^{2k}} = \frac{1}{2}\frac{B_{2k}}{(2k)!} \quad (75)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = (-1)^{k+1}(2\pi)^{2k}\frac{B_{2k}}{2(2k)!} = \zeta(2k) \quad (76)$$

This is the final result for the **Riemann Zeta function of even integers**. Up to now, there is no closed form for odd integers known to man. It also follows (since we have a sum of non negative numbers) that $B_{2k}(-1)^{k+1} \geq 0$.

2.4 Application

The first few results of the Riemann Zeta function are the following:

$$\underbrace{\zeta(2)}_{k=1} = \sum_{n=1}^{\infty}\frac{1}{n^2} = (-1)^2(2\pi)^2\frac{B_2}{2(2)!} = \pi^2 B_2 = \frac{\pi^2}{6} \quad (77)$$

$$\underbrace{\zeta(4)}_{k=2} = \sum_{n=1}^{\infty}\frac{1}{n^4} = (-1)^3(2\pi)^4\frac{B_4}{2(4)!} = -\frac{2^3\pi^4}{4!}B_4 = \frac{\pi^4}{90} \quad (78)$$

$$\underbrace{\zeta(6)}_{k=3} = \sum_{n=1}^{\infty}\frac{1}{n^6} = (-1)^4(2\pi)^6\frac{B_6}{2(6)!} = \frac{2^6\pi^6}{2\times 6!}B_6 = \frac{\pi^6}{945} \quad (79)$$

Further studies on and special application of the Riemann Zeta function:

$$\underbrace{\sum_{n=0}^{\infty}\frac{1}{(2n+2)^p}}_{\text{even numbers}} = \frac{1}{2^p}\left(\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots\right) = \frac{1}{(1\times 2)^p}+\frac{1}{(2\times 2)^p}+\frac{1}{(3\times 2)^p}+\frac{1}{(4\times 2)^p}+\ldots \quad (80)$$

$$= \frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{6^p}+\frac{1}{8^p}+\ldots = \frac{1}{2^p}\sum_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{2^p}\zeta(p) \quad (81)$$

$$\zeta(p) = \sum_{n=1}^{\infty}\frac{1}{n^p} = \underbrace{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^p}}_{\text{odd numbers}} + \underbrace{\sum_{n=0}^{\infty}\frac{1}{(2n+2)^p}}_{\text{even numbers}} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^p} + \frac{1}{2^p}\sum_{n=1}^{\infty}\frac{1}{n^p} \quad (82)$$

$$\Rightarrow \quad \zeta(p) - \frac{1}{2^p}\zeta(p) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^p} = \frac{2^p-1}{2^p}\zeta(p) \quad (83)$$

$$\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\frac{1}{5^p} = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^p}-\sum_{n=0}^{\infty}\frac{1}{(n+1)^p}=\left(2\left(\frac{2^p-1}{2^p}\right)-1\right)\zeta(p) \quad (84)$$

$$= \left(\frac{2^p-2}{2^p}\right)\zeta(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^p} \quad (85)$$

Summarizing we have also found the following infinite series:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^p} = \frac{2^p-2}{2^p}\zeta(p) \quad (86)$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^p} = \frac{2^p-1}{2^p}\zeta(p) \quad (87)$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+2)^p} = \frac{1}{2^p}\zeta(p) \quad (88)$$

Note that one can not put $p=1$ in equation (86) to obtain a result for $\zeta(1)$ in dependnece on $\ln(2)$ since the Riemann zeta function has a singularity at this point due to its reduction to the diverging harmonic series there.

2.5 Euler Product Formula

A similar arguementation as in equation (83) will help to prove the astonishing Euler product represantation formula of the Riemann Zeta function.
First we remove all elements that have a factor of $\frac{1}{2^p}$ or in other words we remove all numbers n in the Zetafunction that have the factor 2. To do so we substract equation (90) from (89):

$$\zeta(p) = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\dots \quad (89)$$

$$\frac{1}{2^p}\zeta(p) = \frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{6^p}+\frac{1}{8^p}+\frac{1}{10^p}+\frac{1}{12^p}+\dots \quad (90)$$

$$\Rightarrow$$

$$\left(1-\frac{1}{2^p}\right)\zeta(p) = \frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\frac{1}{9^p}+\frac{1}{11^p}+\dots \quad (91)$$

Now we proceed to remove all elements of *that* reduced form which have a factor of $\frac{1}{3^p}$ or in other words we further remove all numbers n in the Zetafunction that have the factor 3. (Substract equation (93) from (92))

$$\left(1-\frac{1}{2^p}\right)\zeta(p) = \frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\frac{1}{9^p}+\frac{1}{11^p}+\dots \quad (92)$$

$$\frac{1}{3^p}\left(1-\frac{1}{2^p}\right)\zeta(p) = \frac{1}{3^p}+\frac{1}{9^p}+\frac{1}{15^p}+\frac{1}{21^p}+\frac{1}{27^p}+\frac{1}{33^p}+\dots \quad (93)$$

$$\Rightarrow$$

$$\left(1-\frac{1}{3^p}\right)\left(1-\frac{1}{2^p}\right)\zeta(p) = \frac{1}{1^p}+\frac{1}{5^p}+\frac{1}{7^p}+\frac{1}{11^p}+\frac{1}{13^p}+\frac{1}{17^p}+\dots \quad (94)$$

Repeating this process infinitely often with all prime numbers (only they, otherwise we substract something more than once) we remove all n from the Zeta function up to $\frac{1}{1^p}$!

$$\cdots\left(1-\frac{1}{11^p}\right)\left(1-\frac{1}{7^p}\right)\left(1-\frac{1}{5^p}\right)\left(1-\frac{1}{3^p}\right)\left(1-\frac{1}{2^p}\right)\zeta(p) = 1 \quad (95)$$

$$\Rightarrow \zeta(s)=\frac{1}{\prod_{p=prime}(1-p^{-s})}=\prod_{p=prime}\left(1-p^{-s}\right)^{-1} \quad (96)$$

3 The Finite Exponential Sum

In this section we will derive a closed formula for the exponential sum $\sum_{k=0}^{n} k^l$ To do so we will again compare two notions of one and the same object, in this case a finite series.

3.1 First Notion

Here we will bein by interpreting the sum of interest as a geometric series (see Appendix C). f(x) will be the generating function discussed in Section 1.

$$\sum_{k=0}^{n} e^{kx} = \frac{1-e^{x(n+1)}}{1-e^x} = \frac{e^{x(n+1)}-1}{x}\frac{x}{e^x-1} = \frac{e^{x(n+1)}-1}{x} \times f(x) \tag{97}$$

Now we'll have a closer look at the second last factor and use the series definition of the exponential function.

$$\frac{e^{x(n+1)}-1}{x} = \left(\sum_{k=0}^{\infty}\frac{(n+1)^k}{k!}x^k - 1\right)\frac{1}{x} = \left(\sum_{k=1}^{\infty}\frac{(n+1)^k}{k!}x^k\right)\frac{1}{x} \tag{98}$$

$$= \sum_{k=1}^{\infty}\frac{(n+1)^k}{k!}x^{k-1} = \sum_{k=0}^{\infty}\frac{(n+1)^{k+1}}{(k+1)!}x^k \tag{99}$$

Plugging this result back into equation (97) and using the Cauchy product again (see Appendix A) gives:

$$\sum_{k=0}^{n} e^{kx} = \left(\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n\right)\left(\sum_{k=0}^{\infty}\frac{(n+1)^{k+1}}{(k+1)!}x^k\right) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{B_m x^m}{m!}\frac{(n+1)^{l+1-m}}{(l+1-m)!}x^{l-m} \tag{100}$$

$$= \sum_{l=0}^{\infty}\left[\sum_{m=0}^{l} B_m \binom{l+1}{m}(n+1)^{l+1-m}\right]\frac{x^l}{(l+1)!} \tag{101}$$

3.2 Second Notion

For our second notion of the sum of interest, we will use the definition of the exponential function *inside* the sum first.

$$\sum_{k=0}^{n} e^{kx} = \sum_{k=0}^{n}\sum_{l=0}^{\infty} k^l \frac{x^l}{l!} = \sum_{l=0}^{\infty}\left[\sum_{k=0}^{n} k^l\right]\frac{x^l}{l!} \tag{102}$$

3.3 Comparison and Result

Comparing equation (101) and (102), which are expressions for the same finite series $\sum_{k=0}^{n} e^{kx}$ we obtain:

$$\sum_{l=0}^{\infty}\left[\sum_{m=0}^{l} B_m \binom{l+1}{m}(n+1)^{l+1-m}\right]\frac{x^l}{(l+1)!} = \sum_{l=0}^{\infty}\left[\sum_{k=0}^{n} k^l\right]\frac{x^l}{l!} \tag{103}$$

$$\sum_{k=0}^{n} k^l = \frac{1}{(l+1)}\sum_{m=0}^{l} B_m \binom{l+1}{m}(n+1)^{l+1-m} \tag{104}$$

The first few integer values of l in eqn (104) give well known expressions

$$\sum_{k=0}^{n} k^1 = \frac{1}{2}\sum_{m=0}^{1} B_m \binom{2}{m}(n+1)^{2-m} = \frac{n(n+1)}{2} \tag{105}$$

$$\sum_{k=0}^{n} k^2 = \frac{1}{3}\sum_{m=0}^{2} B_m \binom{3}{m}(n+1)^{3-m} = \frac{n(n+1)(2n+1)}{6} \tag{106}$$

$$\sum_{k=0}^{n} k^3 = \frac{1}{4}\sum_{m=0}^{3} B_m \binom{4}{m}(n+1)^{4-m} = \frac{n^2(n+1)^2}{4} \tag{107}$$

APPENDIX

A Cauchy Product

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) \tag{108}$$

Note: If both series on the left converge absolutely, the Cauchy series converges absolutely as well. In this case, its limit is the product of the limits of the series on the left.

B n choose k

for $n, k \in \mathbb{N}$ we define:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \tag{109}$$

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} \tag{110}$$

C The Geometric Series

For $|q| < 1$

$$s_n = 1 + q^1 + q^2 + q^3 + \ldots + q^n \tag{111}$$

$$s_n \times q = q + q^2 + q^3 + \ldots + q^n + q^{n+1} \tag{112}$$

Substraction of equation (112) from (111) gives

$$s_n(1-q) = 1 - q^{n+1} \quad \Rightarrow \quad s_n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{113}$$

$$s_n - s_{n_0} = \sum_{k=n_0}^{n} q^k = \frac{1-q^{n+1}}{1-q} - \frac{1-q^{n_0-1+1}}{1-q} = \frac{q^{n_0}-q^{n+1}}{1-q} \quad \Rightarrow \quad \sum_{k=n_0}^{\infty} q^k = \frac{q^{n_0}}{1-q} \tag{114}$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} z^k = \frac{z}{1-z} \tag{115}$$

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